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NUMERICAL DETERMINATION BY USE OF SPECIAL COMPUTATIONAL DEVICES OF AN INTEGRAL OPERATOR IN THE THEORY OF COMPRESSIBLE FLUIDS

I. Determination of the Coefficients of the Integral Operator by the Use of Punch Card Machines*

By Stefan Bergman and Leonard Greenstone

1. Introduction. The real or imaginary part of an analytic function of a complex variable, say $g(z)$, $z = x + iy$, furnish solutions of Laplace's equation, which in turn can be interpreted as stream-functions, $\psi(x, y)$, or potentials, $\phi(x, y)$, of two-dimensional steady irrotational flows of an ideal incompressible fluid. As has been shown in [2], [4],[1] this simple procedure can be extended to obtain stream-functions of subsonic flows of a *compressible* fluid.

In addition to the physical plane, i.e., the plane in which the motion actually takes place, whose cartesian coordinates we denote by x and y, we sometimes investigate the functions ϕ and ψ in the hodograph plane; i.e., we consider the functions $\phi^{(1)}(u, v) \equiv \phi[x(u, v), y(u, v)]$ and $\psi^{(1)}(u, v) \equiv \psi[x(u, v), y(u, v)]$, where u and v are the cartesian components of the velocity vector. In the incompressible case $\phi^{(1)}$ and $\psi^{(1)}$ again satisfy Laplace's equation, so that by taking the real or imaginary part of an analytic function we obtain functions $\phi^{(1)}$ and $\psi^{(1)}$ as well as ϕ and ψ. Except in certain special cases, it is simpler to work directly in the physical plane. We note that in the following we shall avoid attaching superscript $^{(1)}$ to $\phi^{(1)}$ and $\psi^{(1)}$.

In the case of a compressible fluid, however, the situation is quite different. In the physical plane the potential and the stream-function satisfy very complicated non-linear equations, while in the hodograph and in certain related planes, they satisfy linear ones. In particular, if (in the *subsonic* case) we introduce variables θ and λ, where

$$\lambda = \tfrac{1}{2}\log\frac{1-T}{1+T} + \frac{1}{2h}\log\frac{1+hT}{1-hT}, \qquad h = \left(\frac{k-1}{k+1}\right)^{\frac{1}{2}}, \tag{1.1}$$

$$T = (1 - M^2)^{\frac{1}{2}}, \qquad [2.3.1], [4.48]$$

where θ is the angle which the velocity vector makes with the positive direction of the x-axis, where M is the Mach number,[2] and where k is a constant character-

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[1] Single numbers in brackets refer to the bibliography at the end of this paper. A pair of numbers $[m \cdot n]$ indicate that the equation which it follows is taken from reference $[m]$, where it appears as formula (n).

[2] We have $M = v[1 - \frac{1}{2}(k-1)v^2]^{-\frac{1}{2}}$, where v is the "reduced velocity," $v = \tilde{v}/a_0$, $\tilde{v}$ being the velocity of the fluid and a_0 the velocity of sound at a stagnation point in the fluid.

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istic of the fluid, then the equation for the stream-function assumes the form

$$\frac{\partial^2\psi}{\partial\lambda^2}+\frac{\partial^2\psi}{\partial\theta^2}+4N\frac{\partial\psi}{\partial\lambda}=0,\quad N=-\frac{(k+1)}{8}\frac{M^4}{(1-M^2)^{3/2}} \tag{1.2}$$

[2.3.3],[4.46,4 7]

(We assume that the flow is adiabatic, the pressure p and density ρ being related by an equation of the form

$$p=A+\sigma\rho^k \tag{1.3}$$

[4.22]

where A, σ, and k are constants.)

According to the theory of integral operators, see [1], [3], the procedure of taking the imaginary part of an analytic function $g(z)$ can be generalized in the following manner to obtain solutions of (1.2). Certain functions $Q^{(n)}(2\lambda)$, $n=1, 2, \cdots$, can be determined such that the expression[3]

$$\begin{cases} \psi(\lambda,\theta)\equiv \operatorname{Im}[\mathrm{P}(g)], \qquad \operatorname{Im}=\text{imaginary part of }\cdots, \\ \mathrm{P}(g)=(1-M^2)^{-\frac{1}{4}}[1+\tfrac{1}{2}(k-1)M^2]^{-1/2(k-1)}\times \\ \qquad\left[g(\mathrm{Z})+\sum_{n=1}^{\infty}\frac{(2n)!}{2^{2n}n!}Q^{(n)}(2\lambda)g^{[n]}(\mathrm{Z})\right], \\ g^{[0]}(\mathrm{Z})\equiv g(\mathrm{Z}), \qquad g^{[n]}(\mathrm{Z})=\int_0^{\mathrm{Z}}g^{[n-1]}(\mathrm{Z}')\,d\mathrm{Z}', \qquad \mathrm{Z}=\lambda-i\theta \end{cases} \tag{1.4}$$

[4.66], [5.97]

where g is an arbitrary analytic function of one complex variable which is regular at the origin, is a solution of (1.2), defined for $|\theta|<|\lambda|\sqrt{3}$, $\lambda<0$, and can therefore be interpreted as the stream-function (in the λ, θ plane) of a possible subsonic flow of a compressible fluid.

The transition to the physical plane, i.e., the representation of the stream-function ψ as a function of x and y, is almost immediate. The stream-lines of the flow under consideration in the physical plane are given in implicit form by the expressions:

$$\begin{cases} x=\displaystyle\int\frac{1}{\rho}\left\{\left[-\frac{(1-M^2)^{\frac{1}{2}}\cos\theta}{v}\psi_\theta-\frac{\sin\theta}{v}\psi_\lambda\right]d\lambda\right. \\ \qquad\left.+\left[\frac{(1-M^2)^{\frac{1}{2}}\cos\theta}{v}\psi_\lambda-\frac{\sin\theta}{v}\psi_\theta\right]d\theta\right\} \\ y=\displaystyle\int\frac{1}{\rho}\left\{\left[-\frac{(1-M^2)^{\frac{1}{2}}\sin\theta}{v}\psi_\theta+\frac{\cos\theta}{v}\psi_\lambda\right]d\lambda\right. \\ \qquad\left.+\left[\frac{(1-M^2)^{\frac{1}{2}}\sin\theta}{v}\psi_\lambda+\frac{\cos\theta}{v}\psi_\theta\right]d\theta\right\} \\ \psi(\lambda,\theta)=\text{constant} \end{cases} \tag{1.5}$$

[4.136] and [4.45]

[3] We note that in [2] and [4] the formula (1.4) has been proved under the assumption that N is replaced by certain expressions N_m approximating N, each N_m satisfying certain

The above formulae enable us to generate from an analytic function g of a complex variable a subsonic flow pattern of a compressible fluid.[4] Since the $Q^{(n)}$ are independent of the flow, it is useful to compute these functions once and for all. The determination of these functions requires a considerable amount of computation, so that it is convenient to use special computational devices such as punch card machines, automatic controlled sequence computers, etc. In order, however, to reduce these computations to a form which permits the use of an automatic computing device, it is useful to transform the equations defining the functions $Q^{(n)}$ into another form, as well as to carry out certain other preliminary steps. This task constitutes the object of the present paper.

2. A modified recurrence formula for $Q^{(n)}$'s. According to [2] and [4], the $Q^{(n)}$ are related by the recurrence formula

$$(2.1)\quad \begin{cases} Q^{(n)} = -\dfrac{1}{2n-1}\left[Q_\lambda^{(n-1)} + 4\displaystyle\int_{-\infty}^{\lambda} FQ^{(n-1)}\,d\lambda\right], \\ \qquad\qquad n = 2, 3, \cdots \qquad [4.84],\ [5.107], \\ Q^{(1)} = -4\displaystyle\int_{-\infty}^{\lambda} F\,d\lambda, \qquad Q^{(n)}(-\infty) = 0, \qquad n = 1, 2, \cdots \end{cases}$$

F itself is given by

$$(2.2)\quad F = \frac{1}{64}(1+k)[5(1+k)T^{-6} - 12kT^{-4} + 2(3k-7)T^{-2} + 4(2+k) - (3k-1)T^2] \qquad [4.71],\ [5.19]$$

where T and λ are connected by the relation (1.1).

As is seen from (2.1) the determination of every new $Q^{(n)}$ involves a differentiation and an integration of the previous $Q^{(n)}$. If the $Q^{(n)}$ have to be determined numerically, then the derivatives would have to be replaced by differences and the integrals by sums. While the second of these operations is feasible and does give fairly good numerical results, the same can not be said for the first. Consequently, in preparing numerical tables it is necessary to arrange our formulas in such a way that we can avoid replacing the derivatives by finite differences. We shall discuss this question in Section 3.

If the derivatives have already been determined satisfactorily at each stage of the determination of $Q^{(n)}$, $n = 1, 2, 3, \cdots$, an integration has to be

---

conditions insuring the convergence of the series (1.4). In [5], see in particular Appendix I, it has been proved subsequently that these conditions hold also for the actual N and that the integration interval of (a,λ) can be taken as $(-\infty, \lambda)$.

[4] Another problem, of course, is that of determining how to choose $g^{[0]}$ in (1.1) in order to obtain in the physical plane a flow of the desired form. This question is discussed in [4] and other publications mentioned in [5] and we shall not concern ourselves with it in the present paper.

performed. Obviously, by replacing the integration by a finite sum we shall at each stage commit an error; these errors will accumulate, inasmuch as the previous $Q^{(n)}$'s are inexact in themselves. It is therefore of importance that the recurrence formulas (2.1) can be replaced by other formulas which although they are more complicated than (2.1) require integration only in the case where n is odd. If n is even, the corresponding $Q^{(n)}$ can be determined from the previous $Q^{(n)}$'s and their derivatives without any integration. These formulas can be obtained from the following

THEOREM. *For every p, $1 \leqq p < n$, the relation*

$$
\begin{aligned}
Q^{(n)} &= -\frac{1}{2n-1} Q_\lambda^{(n-1)} \\
&\quad - \sum_{\nu=1}^{p-1} (-1)^\nu \frac{(2\nu-1)\cdots 5.3.1}{(2n-2\nu+1)\cdots(2n-3)(2n-1)} \times \\
&\qquad \left[Q^{(n-\nu)} Q^{(\nu)} + \frac{1}{2n-2\nu-1} (Q_\lambda^{(n-\nu-1)} Q^{(\nu)} - Q^{(n-\nu-1)} Q_\lambda^{(\nu)}) \right] \\
&\quad - (-1)^p \frac{(2p-1)\cdots 5.3.1}{(2n-2p+1)\cdots(2n-3)(2n-1)} \int_{-\infty}^{\lambda} Q_\lambda^{(p)} Q^{(n-p)} \, d\lambda
\end{aligned} \tag{2.3}
$$

holds.

REMARK. The symbol $\sum_{\nu=1}^{0}$ which we obtain for $p = 1$ is to be interpreted as zero, i.e., the corresponding term does not appear.

PROOF. The proof proceeds by induction on p. For $p = 1$, we have

$$
Q^{(n)} = -\frac{1}{2n-1} Q_\lambda^{(n-1)} + \frac{1}{2n-1} \int_{-\infty}^{\lambda} Q_\lambda^{(1)} Q^{(n-1)} \, d\lambda \tag{2.4}
$$

which is the relation (2.1), since $Q_\lambda^{(1)} = -4F$.

In order to obtain the formula (2.3) for $p = \kappa + 1$, let us assume that the formula (2.3) has been proved for all p, $p = 1, 2, \cdots, \kappa$. We at first integrate by parts $\int_{-\infty}^{\lambda} Q_\lambda^{(\kappa)} Q^{(n-\kappa)} d\lambda$ and using (2.1) replace it by $Q^{(\kappa)} Q^{(n-\kappa)} - \int_{-\infty}^{\lambda} Q_\lambda^{(n-\kappa)} Q^{(\kappa)} d\lambda$. By (2.1) we have

$$
(2n-2\kappa-1) Q_\lambda^{(n-\kappa)} = -Q_{\lambda\lambda}^{(n-\kappa-1)} + Q_\lambda^{(1)} Q^{(n-\kappa-1)} \tag{2.5}
$$

and therefore

$$
\begin{aligned}
-\int_{-\infty}^{\lambda} Q_\lambda^{(n-\kappa)} Q^{(\kappa)} \, d\lambda &= \frac{1}{2n-2\kappa-1} \int_{-\infty}^{\lambda} Q_{\lambda\lambda}^{(n-\kappa-1)} Q^{(\kappa)} \, d\lambda \\
&\quad - \frac{1}{2n-2\kappa-1} \int_{-\infty}^{\lambda} Q_\lambda^{(1)} Q^{(\kappa)} Q^{(n-\kappa-1)} \, d\lambda.
\end{aligned} \tag{2.6}
$$

Performing the integration by parts on both terms on the right-hand side of (2.6), we obtain

$$\frac{1}{2n-2\kappa-1}Q_\lambda^{(n-\kappa-1)}Q^{(\kappa)}-\frac{1}{2n-2\kappa-1}\int_{-\infty}^{\lambda}Q_\lambda^{(n-\kappa-1)}Q_\lambda^{(\kappa)}\,d\lambda$$

$$-\frac{1}{2n-2\kappa-1}Q^{(n-\kappa-1)}\int_{-\infty}^{\lambda}Q^{(\kappa)}Q_\lambda^{(1)}\,d\lambda+ \tag{2.7}$$

$$\frac{1}{2n-2\kappa-1}\int_{-\infty}^{\lambda}Q_\lambda^{(n-\kappa-1)}\left(\int_{-\infty}^{\lambda}Q^{(\kappa)}Q_\lambda^{(1)}\,d\lambda\right)d\lambda$$

Now we collect and combine the second and fourth terms; according to (2.1), $Q_\lambda^{(\kappa)}-\int_{-\infty}^{\lambda}Q^{(1)}Q^{(\kappa)}\,d\lambda=-(2\kappa+1)Q^{(\kappa+1)}$. Further, if we replace the integral in the third term by $(2\kappa+1)Q^{(\kappa+1)}+Q_\lambda^{(\kappa)}$, we then obtain

$$\frac{1}{2n-2\kappa-1}Q_\lambda^{(n-\kappa-1)}Q^{(\kappa)}+\frac{2\kappa+1}{2n-2\kappa-1}\int_{-\infty}^{\lambda}Q_\lambda^{(n-\kappa-1)}Q^{(\kappa+1)}\,d\lambda \tag{2.8}$$

$$-\frac{(2\kappa+1)}{2n-2\kappa-1}Q^{(n-\kappa-1)}Q^{(\kappa+1)}-\frac{1}{2n-2\kappa-1}Q^{(n-\kappa-1)}Q_\lambda^{(\kappa)}.$$

Thus

$$Q^{(\kappa)}Q^{(n-\kappa)}-\int_{-\infty}^{\lambda}Q_\lambda^{(n-\kappa)}Q^{(\kappa)}\,d\lambda=\left[Q^{(\kappa)}Q^{(n-\kappa)}+\frac{1}{2n-2\kappa-1}(Q_\lambda^{(n-\kappa-1)}Q^{(\kappa)}\right. \tag{2.9}$$

$$\left.-Q^{(n-\kappa-1)}Q_\lambda^{(\kappa)})\right]-\frac{(2\kappa+1)}{2n-2\kappa-1}\left(Q^{(n-\kappa-1)}Q^{(\kappa+1)}-\int_{-\infty}^{\lambda}Q_\lambda^{(n-\kappa-1)}Q^{(\kappa+1)}\,d\lambda\right).$$

Substituting (2.9) in (2.3), instead of $\int_{-\infty}^{\lambda}Q_\lambda^{(p)}Q^{(n-p)}\,d\lambda$, we obtain (2.3) for $p=\kappa+1$.

COROLLARY. If n is even, then choosing first p equal to or smaller than $n/2$, say p_1, and then $p=n-p_1$, and adding both expressions, we obtain (replacing p_1 again by p and dividing by 2),

$$\begin{cases} Q^{(n)}=-\dfrac{1}{2n-1}Q_\lambda^{(n-1)}- \\ \left\{\displaystyle\sum_{\nu=1}^{p-1}+\frac{1}{2}\sum_{\nu=p}^{n-p-1}\right\}\left\{(-1)^\nu\dfrac{(2n-1)\cdots 5.3.1}{(2n-2\nu+1)\cdots(2n-3)(2n-1)}\right. \\ \left.\left[Q^{(n-\nu)}Q^{(\nu)}+\dfrac{1}{2n-2\nu-1}(Q_\lambda^{(n-\nu-1)}Q^{(\nu)}-Q^{(n-\nu-1)}Q_\lambda^{(\nu)})\right]\right\} \\ -\dfrac{(-1)^p(2p-1)\cdots 3.1}{2(2n-2p+1)\cdots(2n-3)(2n-1)}Q^{(p)}Q^{(n-p)} \end{cases} \tag{2.10}$$

so that for even n, we have an expression for $Q^{(n)}$ which does not involve any integrals.

REMARK 2.1. If in (2.1) we place $d\lambda$ by $(d\lambda/dT)dT$, the integration for $n = 1$ and $n = 3$ can be carried out in closed form, and we obtain

$$Q^{(1)} = \frac{(k+1)}{8}\left[\frac{1-3k}{k-1}T + \frac{2k}{k+1}T^{-1} - \frac{5}{3}T^{-3} + \frac{8k^2+12k-8}{3(k^2-1)}\right. \tag{2.11}$$
$$\left. + \frac{4}{(k^2-1)^{3/2}}\log\left(\frac{1-h}{1+h}\cdot\frac{1+hT}{1-hT}\right)\right] \qquad [4.107^5],\ [5.109a]$$

$$Q^{(3)} = -\frac{1}{2048}\frac{(k+1)^2}{k-1}\left[\frac{(3k-1)^2}{3}T^3 - \frac{(6k^3+28k^2-58k+16)T}{(k-1)}\right. \tag{2.13}$$
$$+ \frac{(41k^4+96k^3+18k^2-112k-91)(k-1)}{(k+1)^3}T^{-1}$$
$$- \frac{(44k^3+192k^2+236k+80)(k-1)}{3(k+1)^2}T^{-3}$$
$$\left. - \frac{(39k^2-80k-115)(k-1)}{5(k+1)}T^{-5} - 10k(k-1)T^{-7}\right.$$
$$\left. - \frac{25}{9}(k^2-1)T^{-9} + \frac{64h}{(k-1)^2(k+1)^3}\log\left(\frac{h^{-1}-T}{h^{-1}+T}\right)\right]\Bigg|_{T=1}^{T} \qquad [5.109b]$$

Using formula (2.10) we have

$$Q^{(2)} = \tfrac{1}{3}F + \tfrac{1}{6}Q^{(1)2} \tag{2.13}$$ [4.108[

$$Q^{(4)} = -\tfrac{1}{7}Q_\lambda^{(3)} + \tfrac{1}{7}Q^{(3)}Q^{(1)} + \tfrac{1}{35}Q_\lambda^{(2)}Q^{(1)} + \tfrac{4}{35}Q^{(2)}F - \tfrac{3}{70}Q^{(2)2} \tag{2.14}$$ [5.118]

$$Q^{(6)} = -\tfrac{1}{11}Q_\lambda^{(5)} + \tfrac{1}{11}[Q^{(5)}Q^{(1)} + \tfrac{1}{5}(Q_\lambda^{(4)}Q^{(1)} - Q^{(4)}Q_\lambda^{(1)})]$$
$$- \frac{3.1}{9.11}[Q^{(4)}Q^{(2)} + \tfrac{1}{7}(Q_\lambda^{(3)}Q^{(1)} - Q^{(3)}Q_\lambda^{(1)})] - \frac{5.3.1}{2.7.9.11}Q^{(3)2}. \tag{2.15}$$

3. The evaluation of the derivatives of $Q^{(n)}$. It is obvious from formula (2.1) that the derivatives $Q^{(n)}$ can be represented as combinations of $Q^{(p)}$'s, $p < n$ and the derivatives of F.

Examples.

$$Q_\lambda^{(2)} = \tfrac{4}{3}F_\lambda - \tfrac{4}{3}FQ^{(1)} \tag{3.1}$$

$$Q_\lambda^{(3)} = -\tfrac{4}{15}F_{\lambda\lambda} + \tfrac{4}{15}F_\lambda Q^{(1)} - \tfrac{16}{15}F^2 - \tfrac{4}{5}FQ^{(2)} \tag{3.2}$$

$$Q_\lambda^{(4)} = \tfrac{4}{105}F_{\lambda\lambda\lambda} - \tfrac{4}{105}F_{\lambda\lambda}Q^{(1)} + \tfrac{32}{105}F_\lambda F$$
$$- \frac{16F^2}{105}Q^{(1)} + \tfrac{4}{35}F_\lambda Q^{(2)} - \tfrac{4}{7}FQ^{(3)}. \tag{3.3}$$

---

[5] Note that in [4.107] the last term $\frac{4}{(k^2-1)k^2-1}\log\left(1+\frac{k-1}{k+1}T\right)\Big/\left(1-\frac{k-1}{k+1}T\right)$ has to be replaced by $\frac{4}{(k^2-1)\sqrt{k^2-1}}\lg\left[\left(1-\sqrt{\frac{k-1}{k+1}}T\right)\Big/\left(1+k\sqrt{\frac{k-1}{k+1}}T\right)\right]$.

Thus evaluating the derivative of $Q^{(n)}$ if the $Q^{(p)}$'s ($p = 1, 2, \cdots n-1$) are already known, and it is only necessary to determine the derivatives of F. Since

$$\frac{dF}{d\lambda} = \frac{dF}{dT} \Big/ \frac{d\lambda}{dT}, \qquad \frac{dT}{d\lambda} = -\frac{(k+1)}{2} T^{-2} + k - \frac{(k-1)}{2} T^2$$

TABLE I

Values of $A^{(n)}_{sr}$

n	s \ r	0	1	2	3	4	5
	1	15	45	45	58		
	2	0	−54	−108	−54		
	3	−29	−37	61	69		
1	4	0	52	16	−36		
	5	13	−5	−9	9		
	6	0	2	−4	−6		
	7	1	−3	−1	3		
	1	135	540	810	540	135	
	2	0	−648	−1944	−1944	−648	
	3	−280	−600	876	2432	1236	
	4	0	824	952	−1048	−1176	
2	5	158	48	−816	−136	570	
	6	0	−184	120	184	−120	
	7	−16	24	20	−32	−12	
	8	0	8	−24	−8	24	
	9	3	−12	6	12	−9	
	1	810	4050	8100	8100	4050	810
	2	0	−4860	−19,440	−29,160	−19,440	−4860
	3	−1930	−5950	5964	34,292	36,542	12,234
	4	0	7952	16,608	−7296	−32,608	−16,656
	5	1436	1692	−12,384	−13,840	11,940	13,140
3	6	0	−3288	−640	9568	992	−5928
	7	−316	220	2096	−1600	−1780	1380
	8	0	176	−288	−32	288	−144
	9	−6	18	28	−84	−22	66
	10	0	20	−80	40	80	−60
	11	6	−30	36	12	−42	18

these derivatives can be written in terms of T in closed form. A formal computation yields that

$$\frac{d^n F}{d\lambda^n} = \frac{1}{2\chi^{(n)}} \left[\sum_{s=1}^{s=\frac{n+2}{2}n} T^{-3n-s+2s} \left(\sum_{r=0}^{r=n+2} A^{(n)}_{sr} k^r \right) \right] \tag{3.4}$$

TABLE II

Values of $A_{sv}^{(n)}$ *(continuation)*

n	s \ v	0	1	2	3	4	5	6	7
	1	12,150	72,900	182,250	243,000	182,250	72,900	12,150	
	2	0	−87,480	−437,400	−874,800	−874,800	−437,400	−87,480	
	3	−33,380	−135,280	65,764	948,256	1,598,084	1,090,576	273,084	
	4	0	177,208	541,480	78,960	−1,239,920	−1,436,840	−482,232	
	5	31,282	66,116	−349,034	−794,120	93,278	1,029,892	526,362	
	6	0	−108,112	−121,232	433,152	474,368	−335,408	−363,504	
4	7	−11,000	−2,080	138,360	18,080	−286,280	−19,520	155,400	
	8	0	18,512	−14,672	−63,936	52,736	45,424	−38,064	
	9	954	−1,620	−6,026	11,720	710	−10,100	4,362	
	10	0	−248	680	−80	−848	328	168	
	11	−36	144	100	−832	452	688	−516	
	12	0	120	−600	720	240	−840	360	
	13	30	−180	330	−120	−270	300	−90	
	1	109,350	765,450	2,296,350	3,827,250	3,827,250	2,296,350	765,450	109,350
	2	0	−918,540	−5,511,240	−13,778,100	−18,370,800	−13,778,100	−5,511,240	−918,540
	3	−343,010	−1,727,370	−71,082	13,549,790	32,367,930	33,801,570	17,090,770	3,420,618
	4	0	2,230,408	9,005,408	6,301,144	−20,241,344	−41,932,936	−29,581,856	−7,416,408
	5	390,070	1,200,290	−4,948,394	−18,390,974	−9,000,526	24,047,446	30,745,794	10,330,182
	6	0	−1,808,516	−3,764,168	7,513,140	19,484,560	804,580	−18,822,216	−9,611,028
	7	−189,410	−213,410	3,349,486	3,664,606	−8,706,806	−9,395,414	5,608,554	6,006,042
5	8	0	535,472	72,640	−3,204,912	−295,744	5,153,104	244,224	−2,462,544
	9	33,954	−27,426	−503,014	425,222	1,181,254	−1,020,614	−712,194	622,818
	10	0	−38,932	69,608	110,580	−256,784	13,204	187,176	−84,852
	11	−918	2,466	4,162	−17,014	11,390	9,670	−14,634	4,878
	12	0	−312	1,248	−40	−3,584	2,104	2,336	−1,752
	13	−126	630	−14	−3,962	5,334	1,106	−5,194	2,226
	14	0	420	−2,520	4,620	−1,680	−3,780	4,200	−1,260
	15	90	−630	1,530	−1,350	−450	1,710	−1,170	270

where $\chi(n) = n + 5 - \mathrm{E}[(n-1)/2]$, $\mathrm{E}(s)$ being the largest *integer* which equals or is smaller than s. The values of $A_{\nu}^{(n)}$ $(n = 1, 2, 3, 4, 5)$ are given in Tables[6] I and II.

4. Conclusions. In this paper we have indicated a method for the numerical determination of the coefficients of the operator (1.4), i.e., a procedure for the computation of a system of functions $Q^{(n)}$, where each $Q^{(n)}$ is given in terms of the previous function $Q^{(n-1)}$, the formula involving both a differentiation and an integration. (See (2.1).) As was stressed in Section 2, a direct evaluation of the $Q^{(n)}$ has the disadvantage that at each stage we have to perform a numerical differentiation and integration and the errors accumulate, so that a special procedure is needed to handle this question. In our approach this difficulty has been overcome to some extent by proving that there follow from (2.1) modified relations which involve integrations only for odd n. Since the integrations can be carried out in closed form (in terms of T) for $Q^{(1)}$ and $Q^{(3)}$, only three numerical integrations are needed for the determination of the first ten $Q^{(n)}$'s. If, however, we wish to compute considerably more $Q^{(n)}$'s, our formulas become too long and cumbersome and the danger of committing an error greatly increases, particularly because there is no readily apparent law of formation of these functions.

Since for the preparation of tables it is strongly advisable to perform the computation using two completely different methods, it is in our case particularly important to develop another method which will be more appropriate for use with an automatic sequence-controlled calculator and which will permit the determination of a larger number of $Q^{(n)}$'s. Such a procedure will be described in the second part of the present paper.

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[6] We note that the values of $\sum_{\nu=0}^{n+2} A_{\nu}^{(n)} k^{\nu}$ for $k = 1.4$ have been given in [5], see Table, p. 60 in Appendix II. The authors wish to thank Mr. Carl Cohen for his assistance in the preparation of Tables 1 and 2.

TUBULAR LINKAGES

By Michael Goldberg

1. Introduction. Polyhedral linkages, composed of flat plates movably hinged to neighboring plates, can be assembled in various ways. The plates may be arranged in closed chains or closed polyhedra as considered by Bricard [1], or in open polyhedra as considered by Sauer and Graf [2] and Kokotsakis [3]. A survey of these is made in an earlier paper by the author [4].

In the present paper a special form, which may be described as a tubular linkage, is considered.

2. Conditions on tubular linkage joints. Consider two quadrilateral prismatic linkages meeting along a skew quadrilateral joint. Let the developments of these prismatic linkages be similar to those shown in Fig. 1. Let the widths of the faces of one of them be a, b, c, d. Using the normal through the left-most vertex as a reference, let the distances of the vertices along the edges be e, f, g in that order. Let the lengths of the edges of the joint be m, n, r, s in that order.

When this development is folded into a prismatic linkage, the diagonal distances across the linkage vary as the prism is deformed. Let the diagonal distance across the ends of the lengths a and b (and c and d) be t, and the diagonal distance across the lengths a and d (and b and c) be u. Similarly, the diagonal distance across m and n (and r and s) is x, and the diagonal distance across m and s (and n and r) is y. The quadrilateral a, b, c, d is plane, but the quadrilateral m, n, r, s need not be plane.

The following relation holds between the sides a, b, c, d of the plane quadrilateral and its diagonals t, u

$$\begin{aligned}(1)\quad & a^2c^2(b^2+d^2-a^2-c^2+t^2+u^2)+b^2d^2(a^2+c^2-b^2-d^2+t^2+u^2)\\ & +t^2u^2(a^2+b^2+c^2+d^2-t^2-u^2)-(a^2d^2+b^2c^2)t^2-(a^2b^2+c^2d^2)u^2=0\end{aligned}$$

This is obtained by equating to zero the expression for the volume of a tetrahedron in terms of its edges [5]. Since all terms enter as squares, equation (1) can be simplified by letting $a^2 \equiv A$, $b^2 \equiv B$, etc. to obtain equation (2)

$$\begin{aligned}(2)\quad & AC(B+D-A-C+T+U)+BD(A+C-B-D+T+U)\\ & +TU(A+B+C+D-T-U)-(AD+BC)T-(AB+CD)U=0.\end{aligned}$$

The following relations exist between the lengths of the quadrilateral m, n, r, s and the quadrilateral a, b, c, d, where again $M \equiv m^2$, $N \equiv n^2$, etc.

$$\begin{aligned}(3)\quad & M=A+e^2,\ N=B+(f-e)^2,\ R=C+(f-g)^2,\ S=D+g^2,\\ & X=T+f^2,\ Y=U+(e-g)^2.\end{aligned}$$

If the equations (3) are substituted in equation (1), equation (4) is obtained.

$$
\begin{aligned}
&[MR - M(f-g)^2 - Re^2 + e^2(f-g)^2][-M + N - R + S + X \\
&\quad + Y - (-e+f+g)^2] + [NS - S(f-e)^2 - Ng^2 + g^2(f-e)^2][M - N \\
&\quad + R - S + X + Y - (e+f-g)^2] + [XY - X(e-g)^2 - Yf^2 \\
&\quad + f^2(e-g)^2][M + N + R + S - X - Y - (e-f+g)^2] - [MS \\
&\quad + NR - N(f-g)^2 - R(f-e)^2 - Se^2 - Mg^2 + e^2g^2 + (f-e)^2(f-g)^2] \\
&\quad [X - f^2] - [MN + RS - M(f-e)^2 - S(f-g)^2 - Ne^2 - Rg^2 \\
&\quad + e^2(f-e)^2 + g^2(f-g)^2][Y - (e-g)^2] = 0.
\end{aligned} \tag{4}
$$

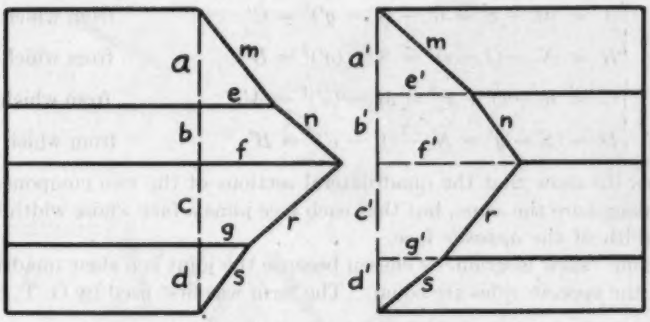

FIG. 1. DEVELOPMENT OF TUBULAR LINKAGE

If equation (4) is rewritten as an equation in the variables X and Y, equation (5) is obtained.

$$
\begin{aligned}
&-XY(X+Y) + (e-g)^2X^2 + f^2Y^2 + XY[M + N + R + S + 2e(f \\
&\quad - g) + 2g(f-e)] + X[(M-N)(R-S) - f^2(M - N - R + S) \\
&\quad - (e-g)^2(M + N + R + S) - 2f\{g(M-N) - e(R-S)\} - 2efg(e \\
&\quad - f + g) + (e-g)^2(e-f+g)^2 + e^2g^2 - (f-e^2)(f-g)^2] + Y[(M \\
&\quad - S)(R-N) - f^2(M + N + R + S) + (N-R)(e^2-g^2) - (S \\
&\quad - M)\{(f-g)^2 - (f-e)^2\} - 2efg(e-f+g) + f^2(e-f+g)^2 + 2e^2g^2 \\
&\quad - e^2(f-e)^2 - g^2(f-g)^2] + k = 0.
\end{aligned} \tag{5}
$$

The parameters e, f, g may sometimes be replaceable by other parameters e', f', g' without changing the values of the coefficients of the variables X and Y in equation (5). If this is the case, it follows that $|f'| = |f|$ since the coefficient of Y^2 is f^2. Similarly, it follows that $|e' - g'| = |e - g|$ since the coefficient of X^2 is $(e-g)^2$.

If, now, another prismatic linkage has the same values of M, N, R, S and the same equation (4) between the variables X and Y, but with a different set of values e', f', g' in place of e, f, g, then this linkage can be joined to the first linkage along the edges m, n, r, s to form a movable linkage of eight hinged plates.

Several different types of linkages, fulfilling the foregoing conditions, are obtainable. The following classification is based upon the shape of the quadrilateral joint m, n, r, s.

3. The skew isogram joint. If $m = r$ and $n = s$ (therefore $M = R$ and $N = S$), then equation (4) will give the same relation between X and Y for two sets e, f, g and e', f', g' if $e - g' = f = f' = e' - g$. Therefore, from equations (3), equations (6) are obtained.

$$(6)\quad \begin{cases} A = M - e^2 = R - (f' - g')^2 = C' & \text{from which } a = c'. \\ B = N - (f - e)^2 = S - (g')^2 = D' & \text{from which } b = d'. \\ C = R - (f - g)^2 = M - (e')^2 = A' & \text{from which } c = a'. \\ D = S - g^2 = N - (f' - e')^2 = B' & \text{from which } d = b'. \end{cases}$$

Equations (6) show that the quadrilateral sections of the two component prismatic linkages are the same, but that each face joins a face whose width is equal to the width of the *opposite* face.

The name "skew isogram" is chosen because the joint is a skew quadrilateral in which the *opposite* sides are equal. The term was first used by G. T. Bennett [6].

4. The skew deltoid joint. If $m = s$ and $r = n$ (therefore $M = S$ and $R = N$) then equation (4) will give the same relation between X and Y for two sets e, f, g and e', f', g' if $e' = g, g' = e$ and $f' = f$. Therefore, from equations (3), equations (7) are obtained.

$$(7)\quad \begin{cases} A = M - e^2 = S - (g')^2 = D' & \text{from which } a = d'. \\ B = N - (f - e)^2 = R - (f' - g')^2 = C' & \text{from which } b = c'. \\ C = R - (f - g)^2 = N - (f' - e')^2 = B' & \text{from which } c = b'. \\ D = S - g^2 = M - (e')^2 = A' & \text{from which } d = a'. \end{cases}$$

Equations (7) show that the quadrilateral sections of the two component prismatic linkages are the same, but that each face joins a face whose width is equal to the width of an *adjacent* face.

The name "*skew deltoid*" is chosen because the joint is a skew quadrilateral composed of two pairs of equal sides in the form of *two isosceles triangles* with a common base. The two triangles are not, in general, in the same plane.

5. The skew equilateral joint. The skew isogram joint and the skew deltoid joint have a shape in common. This occurs as a special case of each, namely,

the case in which $m = n = r = s$. However, the two component prismatic linkages may be joined in two ways. The face of width a may be joined to one of width c as in the skew isogram joint, or to one of width b or d as in the case of the skew deltoid joint.

6. The derived plane joints. In each of the foregoing cases the joint was a skew quadrilateral. As a special case of each, the quadrilateral may be plane. These cases are the plane parallelogram joint, the plane deltoid joint and the plane rhombus joint. Models of these plane joints and the skew joints from which they are derived are shown in Figure 2. In the plane cases, $e + g =$

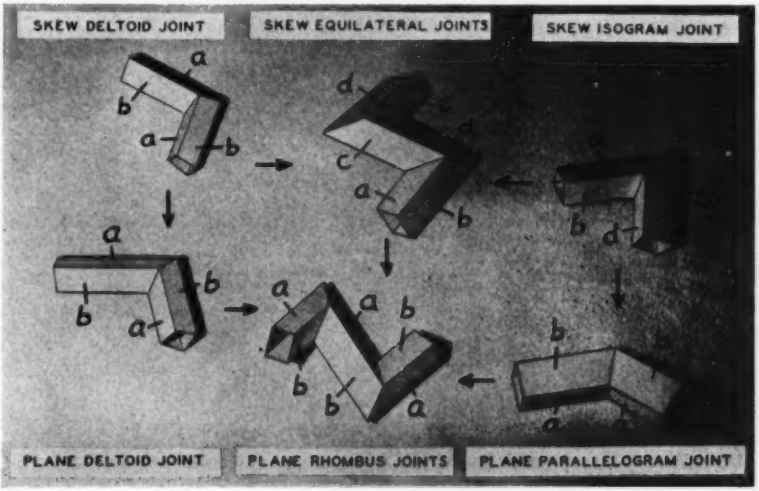

FIG. 2. VARIOUS TUBULAR LINKAGE JOINTS AND RELATIONSHIPS

7. The general plane joints. In addition to the foregoing plane cases, other plane cases are possible. If the joint is a plane parallelogram which remains plane during the motion of the linkage, the relation between X and Y is independent of e, f, g. This relation is given in equation (7).

$$X + Y - 2M - 2N = 0. \tag{7}$$

Therefore, no necessary relation exists between e, f, g and e', f', g' and the component prismatic linkages may have different sections. In fact, any two parallelogram prismatic linkages may be joined to form a composite linkage. Furthermore, the angle between an edge of a joint and the edge of a prismatic linkage may range over a continuous infinity of values. Still further, any selceted face of one prismatic linkage may be joined to any face of the other linkage. Examples of such linkages are shown in Figures 3 and 9.

If the joint is a plane deltoid which remains plane during the motion of the linkage, the relation between X and Y is given by equation (8).

$$X(X + Y - 2M - 2N) + (M - N)^2 = 0. \tag{8}$$

Again, no necessary relation exists between e, f, g and e', f', g' and the component prismatic linkages may have different deltoid sections. In fact, any two prismatic linkages which have different deltoid sections may be joined to form a composite linkage in a continuous infinity of angles. Examples of such linkages are shown in Figures 4 and 10.

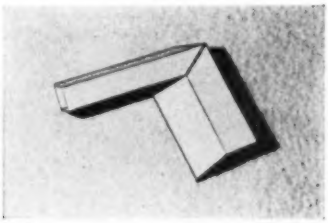

FIG. 3. GENERAL PLANE PARALLELOGRAM JOINT, TWO DIFFERENT PRISMATIC LINKAGES

FIG. 4. CENTRAL PLANE DELTOID JOINT, TWO DIFFERENT PRISMATIC LINKAGES

8. The Bricard deformable octahedron symmetric about a line. Several of the foregoing tubular linkages are derivable from the celebrated deformable octahedron discovered by Bricard [7]. The one which is symmetric about a line is constructed as follows. Consider a skew isogram $UVU'V'$, shown in Figure 5, in which $UV = U'V'$ and $VU' = V'U$. Its axis of symmetry is formed by joining the midpoints U'' and V'' of the lines UU' and VV'. Then join an arbitrary point P to the vertices of $UVU'V'$ by lines to form a pyramidal linkage of the four triangular plates radiating from P. The open boundary of this movable pyramidal linkage is the skew isogram $UVU'V'$. During the motion the distances UU' and VV' vary, but the line joining their midpoints remains perpendicular to them. Therefore, one can select the point P' symmetric to P about the line $U''V''$ and join it to the vertices of $UVU'V'$. The second pyramidal linkage radiating from P' is symmetric to the one radiating

from P; it has equal plates and equal motions. The combination of the two pyramidal linkages is an octahedral linkage with no free edges.

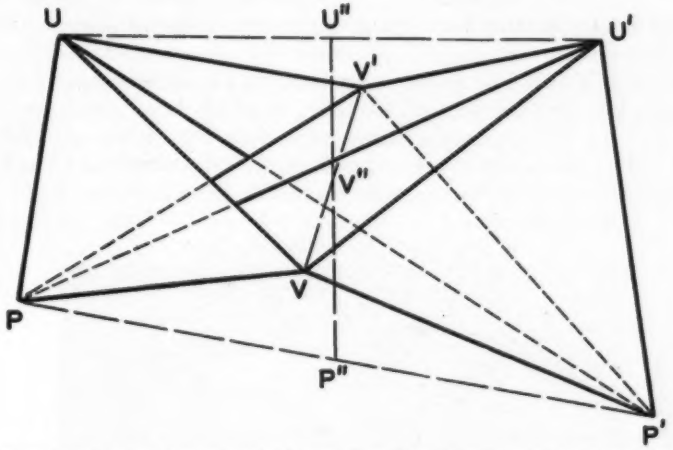

FIG. 5. BRICARD DEFORMABLE OCTAHEDRON SYMMETRIC ABOUT A LINE U''P''

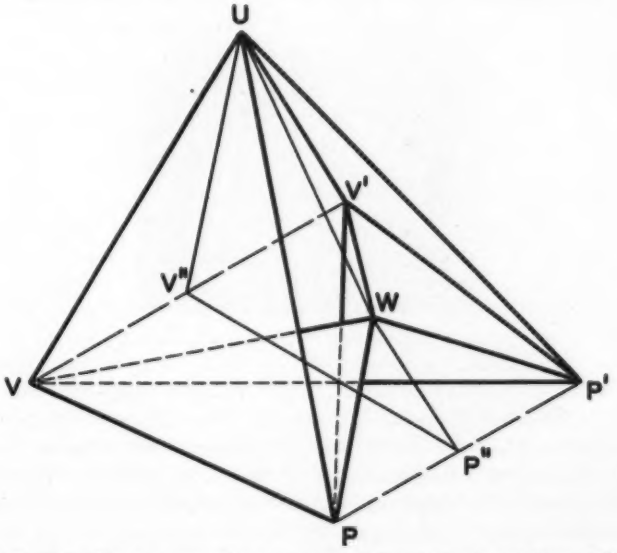

FIG. 6. BRICARD DEFORMABLE OCTAHEDRON SYMMETRIC ABOUT A PLANE UV''P''W

If the points P and P' are moved off to infinity, the two component pyramidal linkages of the deformable octahedron become prismatic linkages, and the

octahedron becomes the tubular linkage with the skew isogram joint described in Section 3, or one of the derived plane joint linkages described in Section 6.

9. The Bricard deformable octahedron symmetric about a plane. This case is based on a skew deltoid $UVWV'$, shown in Figure 6, in which $UV = UV'$ and $VW = V'W$. Let V'' be the midpoint of the variable distance VV'. Then the plane which contains the point V'' and the line UW is the plane of symmetry of the deltoid. Now join an arbitrary point P to the vertices of $UVWV'$ by lines to form a pyramidal linkage of the four triangular plates radiating from P. The open boundary of this movable pyramidal linkage is the skew deltoid $UVWV'$. During the motion the distances UW and VV' vary, but the deltoid

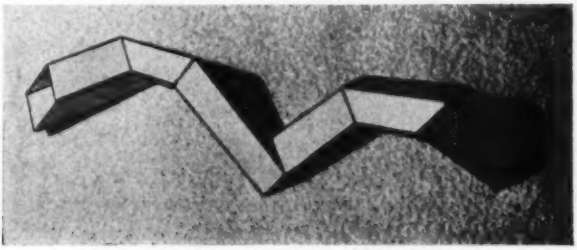

FIG. 7. HELICAL CHAIN TUBULAR LINKAGE

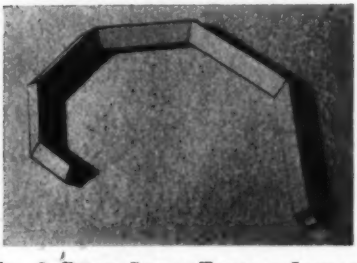

FIG. 8. PLANE SPIRAL TUBULAR LINKAGE

is always symmetric about a plane. Therefore, one can select the point P' symmetric to the point P about this plane of symmetry, and join it by lines to the vertices of $UVWV'$. The second pyramidal linkage radiating from P' is symmetric to the one radiating from P; it has equal plates and equal motions. The combination of the two pyramidal linkages is an octahedron with no free edges.

If the points P and P' are moved off to infinity, the two component pyramidal linkages of the deformable octahedron become prismatic linkages. The octahedron becomes the tubular linkage with the skew deltoid joint described in Section 4 or one of the derived plane joint linkages described in Section 6.

There is another type of Bricard deformable octahedron, namely, the doubly collapsible case. The tubular linkage derived from this case, however, is merely a derived plane joint case of Section 6.

In the Bricard octahedra, shown in Figures 5 and 6, the faces are interpenetrating. However, interpenetration may be avoided by extending one of the pyramidal linkages beyond the joint on the side away from P and removing that portion of the pyramidal linkage on the side toward P. The same remark applies in the case of the tubular linkages derived from them.

It should be noted that the general plane joint linkages, desribed in Section 7, are not derivable from the Bricard octahedra.

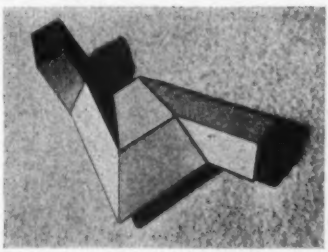

FIG. 9. PLANE PARALLELOGRAM JOINTS, VARIOUS SECTIONS

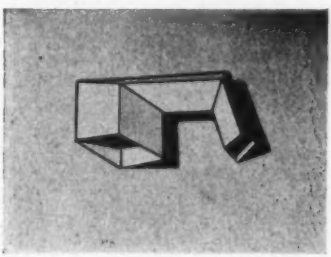

FIG. 10. PLANE DELTOID JOINTS, VARIOUS SECTIONS

10. Chain combinations. Just as two prismatic linkages can be joined to form a tubular linkage of eight plates, any number, say n, of prismatic linkages can be arranged in a chain to form a tubular linkage of $4n$ plates. These may take an infinite variety of shapes. Figures 7 to 11 show several models of such examples. Figure 7 is a succession of equal tubular linkages arranged in the form of a helix which collapses in two distinct positions. Figure 8 is a plane spiral form and it approximates the action of the Bourdon pressure gauge. Figures 9 and 10 are composed of prismatic linkages of various sections with plane joints. Figure 11 is a random mixture of various types of skew deltoid and skew isogram joints. Note that each section in Figure 11 consists of faces of widths a, b, c, d in that order. Note also, that across an isogram joint the

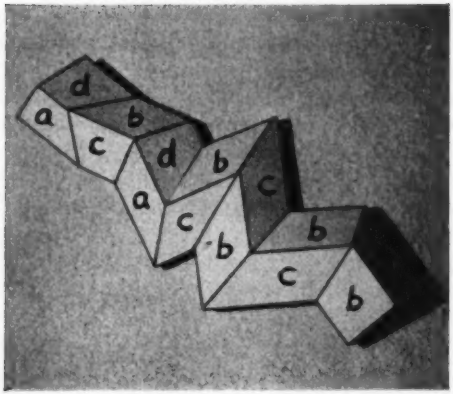

FIG. 11. COMBINED SKEW DELTOID AND SKEW ISOGRAM JOINTS

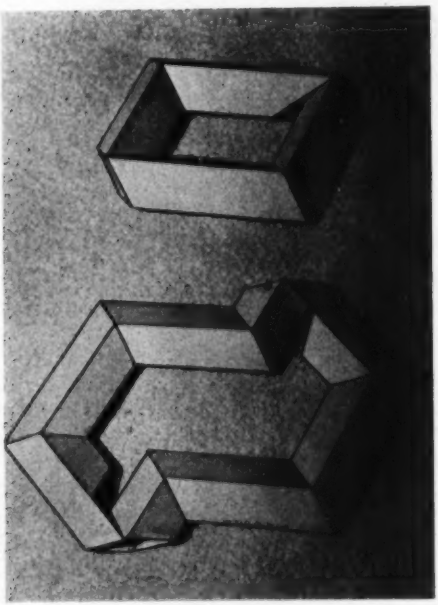

FIG. 12. PLANE POLYGON TOROIDAL LINKAGES

face of width a adjoins a face of width c, while the face of width d adjoins a face of width b. Across a deltoid joint the face of width b adjoins a face of width c and below, hidden from view, a face of width a adjoins a face of width d.

11. Toroidal linkages. Each of the foregoing tubular linkages is an open chain of prismatic linkages. Closed chains are also possible and these are named toroidal linkages. The component prismatic linkages can be arranged in the form of plane or skew polygons as shown in Figures 12 and 13. However, the only ones known have interpenetrating faces similar to the Bricard deformable octahedron. For this reason, working models of these linkages must have portions of the faces removed to permit motion.

Cauchy showed that a closed convex polyhedron linkage is rigid [8]. Bricard's

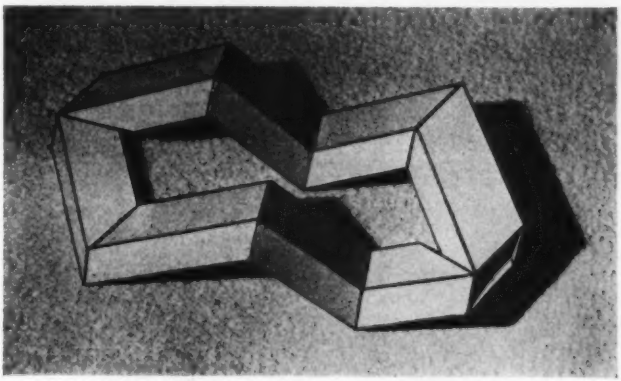

FIG. 13. SKEW POLYGON TOROIDAL LINKAGE

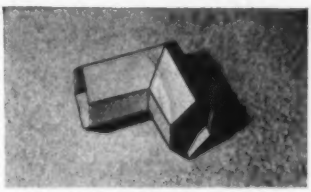

FIG. 14. HEXAGONAL PRISM LINKAGE

deformable octahedron was the first example of a closed non-convex polyhedron linkage that is movable continuously. An open plate linkage can be considered a closed polyhedron if both sides of each plate are considered as bounding faces of a polyhedron of zero volume. The two sides of a plate are a pair of collapsed faces of the polyhedron. Similarly, a chain of rigid polyhedra which are successively hinged along edges can be considered as a single polyhedron in which pairs of edges have collapsed upon each other at the hinges [9]. In these degenerate cases of polyhedral linkages the faces of any separable polyhedron or plate have no relative motion during the motion of the complete polyhedron. However, it is still an open question whether a deformable closed non-convex

polyhedron linkage can be made without interpenetrating faces, collapsed edges or collapsed faces.

12. Generalizations. More general types of tubular linkages are obtainable by employing prisms which have more than four lateral faces. An example is shown in Figure 14 of a hexagonal prism linkage which is obtained from the quadrilateral type by separating the quadrilateral linkage across a diagonal and inserting a pair of parallel and equal faces. An example is shown in Figure 15 of an octagonal prism linkage formed by combining two quadrilateral prism linkages. An example of a network of prismatic linkages is given in a paper by Coxeter [10]. Other generalizations along this line of attack should be obtainable. No attempt has been made to exhaust the possibilities. Further investigations should be fruitful.

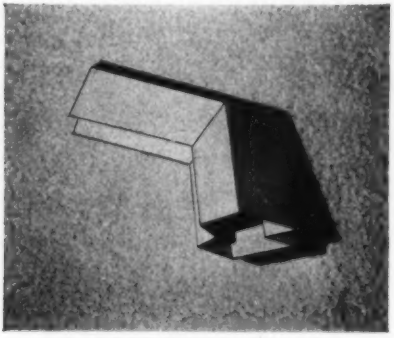

FIG. 15. OCTAGONAL PRISM LINKAGE

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BUREAU OF ORDNANCE NAVY DEPARTMENT

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ON SOME SERIES OF BESSEL FUNCTIONS

By L. Infeld, V. G. Smith and W. Z. Chien

In a study of radiation from a cylindrical antenna in a rectangular wave guide we were confronted with the series

$$\sum_{m=0}^{\infty} (-1)^m Y_0(mx). \tag{1}$$

As the method and result of our solution is of some mathematical interest we shall give here a brief account of it.

Consider the more general problem, to find

$$2\pi \sum_{m=1}^{\infty} \cos(2\pi km) H_0^{(2)}(2\pi z\sqrt{m^2+\epsilon^2}), \qquad 0 \leqslant k < 1. \tag{2}$$

Obviously (1) corresponds to $1/2\pi$ times the imaginary part of (2) for $k = \frac{1}{2}$, $\epsilon = 0$ and $2\pi z = x$.

We shall first prove: *Lemma I*

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-\epsilon\sqrt{x^2-z^2}}}{\sqrt{x^2-z^2}} \cos(tx)\, dx = \sqrt{\frac{\pi}{2}}\, e^{-i(\pi/2)} H_0^{(2)}(z\sqrt{t^2+\epsilon^2}). \tag{3}$$

The proof of (3) follows from a known formula:[1]

$$\int_0^{\infty} J_\mu(xt) \cdot \frac{K_\nu(\epsilon\sqrt{x^2-z^2})}{(x^2-z^2)^{\nu/2}}\, x^{\mu+1}\, dx = \frac{\pi}{2}\, e^{-i\pi(\nu-\mu-1)}\, \frac{t^\mu}{\epsilon^\nu} \left(\frac{\sqrt{\epsilon^2+t^2}}{z}\right)^{\nu-\mu-1} H_{\nu-\mu-1}^{(2)}(z\sqrt{\epsilon^2+t^2}). \tag{4}$$

This is valid if $R(\nu) < 1, z > 0$. Let us put into (4):

$$\mu = -\tfrac{1}{2}, \qquad \nu = \tfrac{1}{2}. \qquad J_{-\frac{1}{2}}(z) = \sqrt{2/\pi z}\cos z, \qquad K_{\frac{1}{2}}(z) = \sqrt{\pi/2z}\, e^{-z}. \tag{5}$$

Then equation (3) follows immediately. From Lemma I we deduce:

A. The Fourier transform of

$$f(x) = \frac{\exp(-\epsilon\sqrt{x^2-z^2})}{\sqrt{x^2-z^2}} \quad \text{is} \quad g(t) = \sqrt{\pi/2}\, e^{-i(\pi/2)} H_0^{(2)}(z\sqrt{t^2+\epsilon^2}) \tag{6}$$

because

$$g(t) = \sqrt{2/\pi} \int_0^{\infty} f(x) \cos(xt)\, dx. \tag{7}$$

B. By the reciprocal property of the Fourier transformation we obtain:

$$\int_0^{\infty} H_0^{(2)}(z\sqrt{t^2+\epsilon^2}) \cos(xt)\, dt = i \cdot \exp(-\epsilon\sqrt{x^2-z^2})/\sqrt{x^2-z^2}. \tag{8}$$

[1] G. N. Watson: *Theory of Bessel Functions*, Camb. Univ. Press. 1922, p. 416, eq. (3). A slight misprint in Watson's formula has been corrected by us.

Lemma II. The Fourier transform of

$$g(t) = \cos(2\pi kt) H_0^{(2)}(z\sqrt{t^2+\epsilon^2}) \tag{9}$$

is

$$f(x) = \frac{i}{\sqrt{2\pi}}\left[\frac{\exp(-\epsilon\sqrt{(x+2k\pi)^2 - z^2})}{\sqrt{(x+2k\pi)^2 - z^2}} + \frac{\exp(-\epsilon\sqrt{(x-2k\pi)^2 - z^2})}{\sqrt{(x-2k\pi)^2 - z^2}}\right]. \tag{10}$$

To prove this lemma we calculate

$$\begin{aligned} f(x) &= \sqrt{2/\pi}\int_0^\infty \cos(xt)\cos(2\pi kt) H_0^{(2)}(z\sqrt{t^2+\epsilon^2})\,dt \\ &= \sqrt{1/2\pi}\int_0^\infty H_0^{(2)}(z\sqrt{\epsilon^2+t^2})\,[\cos(x+2\pi k)t + \cos(x-2\pi k)t]\,dt \end{aligned} \tag{11}$$

Lemma II follows immediately from the last equation and from (8).

We can now perform the summation (2) using Poisson's summation formula:

$$\sqrt{2\pi}\left[\tfrac{1}{2}f(0) + \sum_{n=1}^{\infty} f(2\pi n)\right] = \tfrac{1}{2}g(0) + \sum_{n=1}^{\infty} g(n). \tag{12}$$

This formula is valid if $f(x)$ is integrable in the interval $(0, \infty)$, continuous and steadily decreasing to zero as $x \to \infty$, and if $g(x)$ is the Fourier cosine transform of $f(x)$. We introduce into (12):

$$\begin{cases} g(x) = \cos(2\pi kx) H_0^{(2)}(z\sqrt{x^2+\epsilon^2}) \\ f(x) = \dfrac{i}{\sqrt{2\pi}}\left[\dfrac{\exp(-\epsilon\sqrt{(x+2k\pi)^2 - z^2})}{\sqrt{(x+2k\pi)^2 - z^2}} + \dfrac{\exp(-\epsilon\sqrt{(x-2k\pi)^2 - z^2})}{\sqrt{(x-2k\pi)^2 - z^2}}\right] \end{cases} \tag{13}$$

according to (9) and (10). Replacing z by $2\pi z$ we obtain the required summation formula:

$$\begin{aligned} 2\pi\sum_{m=1}^{\infty} \cos(2\pi km) H_0^{(2)}(2\pi z\sqrt{m^2+\epsilon^2}) &= i\frac{\exp(-2\pi\epsilon\sqrt{k^2 - z^2})}{\sqrt{k^2 - z^2}} \\ &+ i\sum_{n=1}^{\infty}\left[\frac{\exp(-2\pi\epsilon\sqrt{(n+k)^2 - z^2})}{\sqrt{(n+k)^2 - z^2}} + \frac{\exp(-2\pi\epsilon\sqrt{(n-k)^2 - z^2})}{\sqrt{(n-k)^2 - z^2}}\right] - \pi H_0^{(2)}(2\pi\epsilon z). \end{aligned} \tag{14}$$

This formula contains the solution of our problem. The transition $\epsilon \to 0$ is not trivial and has to be performed carefully.

To find this limit we write the right hand side of (14) in three parts as follows:

$$\text{(15a)}\qquad i\sum_{n=1}^{\infty}\left[\frac{\exp\left(-2\pi\epsilon\sqrt{(n+k)^2-z^2}\right)}{\sqrt{(n+k)^2-z^2}}-\frac{\exp\left(-2\pi\epsilon(n+k)\right)}{n+k}\right]$$
$$+\,i\sum_{n=1}^{\infty}\left[\frac{\exp\left(-2\pi\epsilon\sqrt{(n-k)^2-z^2}\right)}{\sqrt{(n-k)^2-z^2}}-\frac{\exp\left(-2\pi\epsilon(n-k)\right)}{n-k}\right]$$

$$\text{(15b)}\qquad i\sum_{n=1}^{\infty}\left[\frac{\exp\left(-2\pi\epsilon(n+k)\right)}{n+k}-\frac{\exp\left(-2\pi\epsilon n\right)}{n}\right]$$
$$+\,i\sum_{n=1}^{\infty}\left[\frac{\exp\left(-2\pi\epsilon(n-k)\right)}{n-k}-\frac{\exp\left(-2\pi\epsilon n\right)}{n}\right]$$

$$\text{(15c)}\qquad 2i\sum_{n=1}^{\infty}\frac{\exp\left(-2\pi\epsilon n\right)}{n}+i\,\frac{\exp\left(-2\pi\epsilon\sqrt{k^2-z^2}\right)}{\sqrt{k^2-z^2}}-\pi H_0^{(2)}(2\pi\epsilon z)$$

As $\epsilon\to 0$, (15a) approaches as a limit,

$$\text{(16a)}\qquad i\sum_{n=1}^{\infty}\left[\left(\frac{1}{\sqrt{(n+k)^2-z^2}}-\frac{1}{(n+k)}\right)+\left(\frac{1}{\sqrt{(n-k)^2-z^2}}-\frac{1}{n-k}\right)\right]$$

while (15b) becomes

$$\text{(16b)}\qquad i\sum_{n=1}^{\infty}\left[\left(\frac{1}{n+k}-\frac{1}{n}\right)+\left(\frac{1}{n-k}-\frac{1}{n}\right)\right]=-i[2\gamma+\psi(k)+\psi(-k)]$$

where γ is Euler's constant 0.577,215,665 $\cdots$ and

$$\psi(k)=\frac{d}{dk}\log\Gamma(1+k)=-\gamma+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+k}\right)$$

and (15c) becomes

$$\text{(16c)}\qquad -\pi+i/\sqrt{k^2-z^2}+2i(\gamma+\log\tfrac{1}{2}z).$$

Inserting these limiting values in (10) we find that

$$2\pi\sum_{m=1}^{\infty}\cos(2\pi km)H_0^{(2)}(2\pi zm)=-\pi+2i\log\tfrac{1}{2}z-i[\psi(k)+\psi(-k)]$$

$$\text{(17)}\qquad +\frac{i}{\sqrt{k^2-z^2}}+i\sum_{n=1}^{\infty}\left[\left(\frac{1}{\sqrt{(n+k)^2-z^2}}-\frac{1}{n+k}\right)+\left(\frac{1}{\sqrt{(n-k)^2-z^2}}-\frac{1}{n-k}\right)\right].$$

Special Cases. (A) Let $k=0$; $\psi(0)=-\gamma$ and consequently

$$\text{(18)}\qquad 2\pi\sum_{m=1}^{\infty}H_0^{(2)}(2\pi mz)=-\pi+\frac{1}{z}$$
$$+\,2i\left[\left(\gamma+\log\frac{z}{2}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n^2-z^2}}-\frac{1}{n}\right)\right].$$

From this we have, separating the real and imaginary parts,

$$\pi \sum_{m=1}^{\infty} J_0(2\pi mz) = -\frac{\pi}{2} + \frac{1}{2z} + \sum_{n=1}^{n_0} \frac{1}{\sqrt{z^2 - n^2}} \tag{19a}$$

$$\pi \sum_{m=1}^{\infty} Y_0(2\pi mz) = -\left(\gamma + \log \frac{z}{2}\right) + \sum_{n=1}^{n_0} \frac{1}{n} - \sum_{n=(n_0+1)}^{\infty} \left(\frac{1}{\sqrt{n^2 - z^2}} - \frac{1}{n}\right) \tag{19b}$$

where n_0 is the greatest integer less than z.

(B) Let $k = \frac{1}{2}$; $\psi(\frac{1}{2}) = -\gamma - 2 \log 2$, $\psi(-\frac{1}{2}) = -\gamma - 2 \log 2 + 2$, and therefore

$$2\pi \sum_{m=1}^{\infty} (-1)^m H_0^{(2)}(2\pi mz) = -\pi + 2i(\gamma + \log 2z) + 4i \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{(2n-1)^2 - (2z)^2}} - \frac{1}{2n-1}\right). \tag{15}$$

Again separating the real and imaginary parts, (z real) and writing z for $2z$, we have

$$\pi \sum_{m=1}^{\infty} (-1)^m J_0(\pi mz) = -\frac{\pi}{2} + 2 \sum_{n=1}^{n_0'} \frac{1}{\sqrt{z^2 - (2n-1)^2}} \tag{20a}$$

$$\pi \sum_{m=1}^{\infty} (-1)^m Y_0(\pi mz) = -(\gamma + \log z) + 2 \sum_{n=1}^{n_0'} \frac{1}{2n-1} - 2 \sum_{n=(n_0'+1)}^{\infty} \left(\frac{1}{\sqrt{(2n-1)^2 - z^2}} - \frac{1}{2n-1}\right) \tag{20b}$$

where n_0' is the greatest integer for which $z^2 > (2n_0' - 1)^2$

Convergence. While in (19b) and (20b) the infinite series on the right are much more rapidly convergent than those on the left, they are too slow for easy computation, especially if z is at all large. To avoid this difficulty the series may be broken into two parts and the radicals of the second part expanded. For example, consider (20b) which may be written,

$$\frac{\pi}{2} \sum_{m=1}^{\infty} (-1)^m Y_0(\pi mz) = -\tfrac{1}{2}(\gamma + \log z) + \sum_{n=1}^{N-1} \frac{1}{2n-1} - \sum_{n=n_0'+1}^{N-1} \frac{1}{\sqrt{(2n-1)^2 - z^2}} - \sum_{n=N}^{\infty} \left(\frac{1}{\sqrt{(2n-1)^2 - z^2}} - \frac{1}{2n-1}\right) \tag{21}$$

where $N \geq n_0' + 1$, but is otherwise an arbitrary integer. The last term of (21) may be expanded and the order of summation interchanged so that it becomes

$$\sum_{p=1}^{\infty} \frac{1 \cdot 3 \cdots (2p-1)}{2 \cdot 4 \cdots (2p)} z^{2p} \sum_{n=N}^{\infty} \frac{1}{(2n-1)^{2p+1}}. \tag{22}$$

But

$$\sum_{n=N}^{\infty} \frac{1}{(2n-1)^{2p+1}} = \left(1 - \frac{1}{2^{2p+1}}\right) S_{2p+1} - \sum_{n=1}^{N-1} \frac{1}{(2n-1)^{2p+1}} \tag{23}$$

where

$$S_{2p+1} = \sum_{n=1}^{\infty} \frac{1}{n^{2p+1}} \tag{24}$$

is a tabulated function.[2]

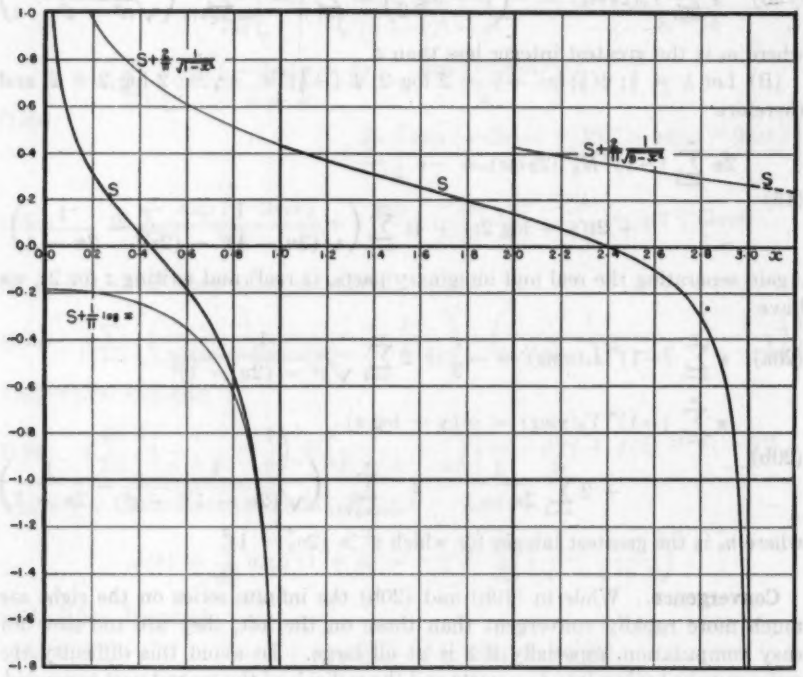

FIG. 1. Values of the function $S = \sum_{m=1}^{\infty} (-1)^m Y_0(m\pi x)$ in the interval $x(0, 3)$. The light curves show the values of the function with one term removed as indicated, thus making an accurate estimate possible near one of the discontinuities.

Hence (21) becomes

$$\frac{\pi}{2} \sum_{m=1}^{\infty} (-1)^m Y_0(m\pi z) = -\tfrac{1}{2}(\gamma + \log z) + \sum_{n=1}^{N-1} \frac{1}{2n-1} - \sum_{n=n_0+1}^{N-1} \frac{1}{\sqrt{(2n-1)^2 - z^2}} - \sum_{p=1}^{\infty} a_p z^{2p} \tag{25}$$

where

$$a_p = \frac{1 \cdot 3 \cdots (2p-1)}{2 \cdot 4 \cdots 2p} \left[\left(1 - \frac{1}{2^{2p+1}}\right) S_{2p+1} - \sum_{n=1}^{N-1} \frac{1}{(2n-1)^{2p+1}} \right]. \tag{26}$$

[2] Integral Calculus, J. Edwards, Vol. II, p. 144, Macmillan.

The choice of N is determined by the fact that if it is chosen too small the power series converges too slowly, while if too large there are too many radical terms to calculate. With a good table[3] of $(1-\beta^2)^{-1/2}$, the radical terms may be computed by interpolation.

This method of computing a_p becomes too laborious for larger values of N and p. For example, when $N = 7$, $p = 5$, fifteen decimal places must be carried

TABLE I

The function $S = \sum_{m=1}^{\infty} (-1)^m Y_0(mx)$

x	S	$S+\frac{1}{\pi}\log_e x$	$S+\frac{2}{\pi}\frac{1}{\sqrt{1-x^2}}$
0.0	$+\infty$	−0.18373 34526	$+\infty$
0.2	0.31477 77147	−0.19752 22840	0.96452 50491
0.4	0.04727 54212	−0.24438 89773	0.74188 45393
0.6	−0.18636 79190	−0.34896 87651	0.60940 67965
0.8	−0.54813 89512	−0.61916 77497	0.51289 40027
1.0(−)	$-\infty$	$-\infty$	0.43521 42278
1.0(+)	0.43521 42278		
1.2	0.36854 38750		
1.4	0.30844 74844		$S+\frac{2}{\pi}\frac{1}{\sqrt{9-x^2}}$
1.6	0.25190 16535		
1.8	0.19637 88834		
2.0	0.13918 51818		0.42389 01992
2.2	0.07657 73314		0.38870 58547
2.4	0.00176 84861		0.35544 61374
2.6	−0.10177 52043		0.32358 42732
2.8	−0.29843 04327		0.29265 61869
3.0(−)	$-\infty$		0.26222 12325
3.0(+)	0.26222 12325		

to yield a_5 to three significant figures. We can then resort to the asymptotic series obtained by the usual method of replacing summation by integration:

$$a_p = \frac{1\cdot 3\cdots(2p-1)}{2\cdot 4\cdots 2p}\left[\frac{1}{4p(2N-1)^{2p}} + \frac{1}{2(2N-1)^{2p+1}}\right.$$

$$+\frac{(2p+1)}{6(2N-1)^{2p+2}} - \frac{(2p+1)(2p+2)(2p+3)}{90(2N-1)^{2p+4}} + \cdots \tag{27}$$

$$\left. + (-1)^{r-1}\frac{2^{2r-1}B_{2r-1}}{\lfloor 2r}\cdot\frac{(2p+1)(2p+2)+\cdots(2p+2r-1)}{(2N-1)^{2p+2r}} + \cdots\right]$$

[3] Miscellaneous Physical Tables, Bureau of Standards, 1941.

where B_{2r-1} is the rth Bernoullian number. Eleven terms of this series ($N = 7$, $p = 5$) give a_5 to seven significant figures within a quarter of a unit in the seventh place.

To estimate the remainder omitted by using only $(P-1)$ terms of the power series we have

$$\sum_{n=N}^{\infty} \frac{1}{(2n-1)^{2p+1}} < \int_{x=N-1}^{\infty} \frac{dx}{(2x-1)^{2p+1}} = \frac{1}{4p(2N-3)^{2p}} \tag{28}$$

so that, if $p > P-1$

$$a_p < \frac{1 \cdot 3 \cdots (2P-1)}{2 \cdot 4 \cdots (2P)} \cdot \frac{1}{4P(2N-3)^{2p}} \tag{29}$$

and the error η is

$$\eta = \sum_{p=P}^{\infty} a_p z^{2p} < \frac{1 \cdot 3 \cdots (2P-1)}{2 \cdot 4 \cdots (2P)} \cdot \frac{1}{4P} \sum_{p=P}^{\infty} \frac{z^{2p}}{(2N-3)^{2p}} \tag{30}$$

or

$$\eta < \frac{1 \cdot 3 \cdots (2P-1)}{2 \cdot 4 \cdots (2P)} \cdot \frac{K^{2P}}{4P(1-K^2)} \tag{31}$$

where

$$K = z/(2N-3) \tag{32}$$

A few trials will tell what values of N and P to choose to give any required degree of accuracy for the greatest value of z required. The series will then do for all smaller values of z but the sudden change of n_0' as z passes through odd integral values must be remembered.

Equation (19b) may be treated in a similar manner.

Table I gives the numerical values of the function $S = \sum_{m=1}^{\infty}(-1)^m Y_0(m\pi x)$ in the interval $0 \leq x \leq 3$. The values were calculated to thirteen decimals and rounded off to ten. The second and third columns give the function with one term removed to make possible interpolation near a discontinuity.

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MATHIEU FUNCTIONS OF FRACTIONAL ORDER

By N. W. McLachlan

1. Introduction. These functions are solutions of the differential equations

$$d^2y/dz^2 + (a \mp 2q \cos 2z)y = 0, \qquad (1, 2)$$

and their modified versions

$$d^2y/dz^2 - (a \mp 2q \cosh 2z)y = 0, \qquad (3, 4)$$

when the parametric point (a, q) lies in a stable region of the (a, q) plane [7, 8]. Equations (1), (2) and (3), (4) are mutually transformable, each to each, if $\pm iz$ is written for z. In references 7, 8, two sets of fractional order solutions of (1) are defined as follows:

$$\left.\begin{matrix} ce \\ se \end{matrix}\right\}_{2n+\beta}(z, q) = \sum_{r=-\infty}^{\infty} A_{2r}^{(2n+\beta)} \begin{matrix} \cos \\ \sin \end{matrix} (2r + \beta)z, \qquad (a_{2n+\beta}) \qquad (5)$$

and

$$\left.\begin{matrix} ce \\ se \end{matrix}\right\}_{2n+1+\beta}(z, q) = \sum_{r=-\infty}^{\infty} A_{2r+1}^{(2n+1+\beta)} \begin{matrix} \cos \\ \sin \end{matrix} (2r + 1 + \beta)z, \qquad (a_{2n+1+\beta}) \qquad (6)$$

where $q > 0$, and $0 < \beta < 1$. $a_{2n+\beta}$ and $a_{2n+1+\beta}$ are the characteristic numbers for the functions of order $(2n + \beta)$, $(2n + 1 + \beta)$, respectively. Their graphs, as functions of q, are depicted in reference 8, Fig. 2. If (5), (6) are substituted into (1) and the coefficient of $\cos (2r + \beta)z$, $\cos(2r + 1 + \beta)z$ equated to zero for $r = -\infty$ to $+\infty$, the recurrence relations

$$[a - (2r + \beta)^2]A_{2r} - q(A_{2r+2} + A_{2r-2}) = 0 \qquad (7)$$

and

$$[a - (2r + 1 + \beta)^2]A_{2r+1} - q(A_{2r+3} + A_{2r-1}) = 0 \qquad (8)$$

are obtained, respectively, in which the superscripts to the A have been omitted. The same relations are obtained by using $\sin (2r + \beta)z$, $\sin (2r + 1 + \beta)z$ in like manner. When $\beta = p/s$, a rational fraction in its lowest terms, the functions in (5), (6) have period $2s\pi$ in z, real, $s \geqslant 2$; if β is irrational they are non-periodic, being bounded in both cases. It may be shown that the above series representations of the functions are absolutely and uniformly convergent in any finite region of the z-plane. Thus writing iz for z in (5), we define the *modified* Mathieu functions of fractional order $(2n + \beta)$ to be

$$\left.\begin{matrix} Ce \\ Se \end{matrix}\right\}_{2n+\beta}(z, q) = \sum_{r=-\infty}^{\infty} A_{2r}^{(2n+\beta)} \begin{matrix} \cosh \\ \sinh \end{matrix} (2r + \beta)z. \qquad (a_{2n+\beta}) \qquad (9)$$

These series are solutions of equation (3), with $q > 0$, $0 < \beta < 1$, and characteristic number $a_{2n+\beta}$. Similar definitions for the modified functions of

order $(2n + 1 + \beta)$ follow from (6) by writing iz for z. The multiplier i is omitted from the sinh series. When $\beta = p/s$, a rational fraction in its lowest terms, the functions have period $2s\pi i$, $s \geqslant 2$.

If in (1) we write $(\frac{1}{2}\pi - z)$ for z, the sign of q is changed and we get equation (2). Making this substitution in (5) yields

$$ce_{2n+\beta}(\tfrac{1}{2}\pi - z, q) = \cos \tfrac{1}{2}\beta\pi \sum_{r=-\infty}^{\infty} (-1)^r A_{2r}^{(2n+\beta)} \cos(2r + \beta)z + \sin \tfrac{1}{2}\beta\pi \sum_{r=-\infty}^{\infty} (-1)^r A_{2r}^{(2n+\beta)} \sin (2r + \beta)z. \tag{10}$$

Since (10) contains *both* the even and odd solutions of (2), its form is undesirable in either analytical work or in applications. This 'duality' may be overcome by a simple artifice. The recurrence relations for the coefficients in the solutions of (2) are those at (7), (8) with $-q$ for q. This change is effected in (7) by writing $(-1)^r A_{2r}$ for A_{2r}, and in (8) by a similar substitution. Hence the solutions of (1) for $q < 0$, i.e. those of (2) with $q > 0$, are *defined* to be

$$\left.\begin{matrix} ce \\ se \end{matrix}\right\}_{2n+\beta}(z, q) = (-1)^n \sum_{r=-\infty}^{\infty} (-1)^r A_{2r}^{(2n+\beta)} \begin{matrix} \cos \\ \sin \end{matrix} (2r + \beta)z. \tag{11}$$

The solutions corresponding to (6), (9) are derived therefrom by writing $(-1)^r$ within the $\sum$ sign and $(-1)^n$ outside it. When $q \to 0$, it may be shown that $A_{2n}^{(2n+\beta)} \to 1$, while all the remaining coefficients tend to zero. Thus the two solutions in (5) reduce to $+\cos(2n + \beta)z$, and $+\sin(2n + \beta)$, respectively. In (11) the multiplier $(-1)^n$ ensures the conventional positive signs for the limiting forms.

2. Representation of functions in §1 in Bessel function product series. In reference 2 it is shown that the series

$$\sum_{r=-\infty}^{\infty} (-1)^r \phi(r + \tfrac{1}{2}\beta) J_r(v_1) J_{r+\beta}(v_2) \tag{1}$$

is a solution of (3) §1, with $v_1 = ke^{-s}$, $v_2 = ke^{s}$, $k = +q^{\frac{1}{2}}$, while ϕ satisfies a recurrence relation of the form (7) §1. We shall now express the functions in §1 in terms of Bessel function products, by aid of (1) above and (7) §1. The latter is a linear difference equation having two independent solutions. Writing $(r + 1)$ for r, dividing throughout by A_{2r} and putting $A_{2r+2}/A_{2r} = v_{2r}$ leads to

$$[a - (2r + 2 + \beta)^2]v_{2r} - q(v_{2r+2}v_{2r} + 1) = 0, \tag{2}$$

so

$$v_{2r+2} + 1/v_{2r} = [a - (2r + 2 + \beta)^2]q^{-1}. \tag{3}$$

Then as $r \to \pm\infty$, $v_{2r+2} + 1/v_{2r} \to -\infty$ monotonically, and it is evident that v_{2r} cannot, (a) oscillate boundedly, (b) tend to a unique *finite* limit other than zero. Hence either $v_{2r} \to 0$, or $|v_{2r}| \to \infty$, so one solution of (7) §1 tends to

zero, but the other to infinity as $r \to \pm\infty$. For convergence of the series (1) above, and those in §1, it may be shown that the solution of (2) which tends to zero is admissible, whereas the other is not. Thus it follows that $\phi(r + \frac{1}{2}\beta)$ must be a constant multiple of $A_{2r}^{(2n+\beta)}$. Hence

$$\sum_{r=-\infty}^{\infty} (-1)^r A_{2r} J_r(v_1) J_{r+\beta}(v_2) \tag{4}$$

is a solution of (3) §1, which is neither odd nor even in z. Now (3) §1 is unaltered if we write $-z$ for z, so we infer that

$$\sum_{r=-\infty}^{\infty} (-1)^r A_{2r} J_r(v_2) J_{r+\beta}(v_1) \tag{5}$$

is also a solution. Adding and subtracting (4), (5) yields the two series, even and odd, respectively, in z

$$\sum_{r=-\infty}^{\infty} (-1)^r A_{2r} [J_r(v_1) J_{r+\beta}(v_2) \pm J_{r+\beta}(v_1) J_r(v_2)]. \tag{6}$$

It may be demonstrated that (6) is absolutely and uniformly convergent in any finite region of the z-plane, so if we write $-iz$ for z, we get

$$\sum_{r=-\infty}^{\infty} (-1)^r A_{2r} [J_r(v_1') J_{r+\beta}(v_2') \pm J_{r+\beta}(v_1') J_r(v_2')], \tag{7}$$

with $v_1' = ke^{iz}$ and $v_2' = ke^{-iz}$. The two series in (7) are even and odd solutions of (1) §1. For a given $a = a_{2n+\beta}$, this equation is known to have only one even solution and a corresponding odd one. We infer, therefore, that the two series in (7) are constant multiples of $ce_{2n+\beta}(z, q)$ and $se_{2n+\beta}(z, q)$, respectively. Thus we have

$$\left.\begin{matrix} ce \\ se \end{matrix}\right\}_{2n+\beta} (z, q) = \left.\begin{matrix} C \\ iS \end{matrix}\right\} \sum_{r=-\infty}^{\infty} (-1)^r A_{2r} [J_r(v_1') J_{r+\beta}(v_2') \pm J_{r+\beta}(v_1') J_r(v_2')], \tag{8}$$

where

$$C = ce_{2n+\beta}(0, q)/2 \sum_{r=-\infty}^{\infty} (-1)^r A_{2r} J_r(k) J_{r+\beta}(k), \tag{9}$$

and

$$S = se'_{2n+\beta}(0, q)/2k \sum_{r=-\infty}^{\infty} (-1)^r A_{2r} [J_r(k) J'_{r+\beta}(k) - J_{r+\beta}(k) J'_r(k)]. \tag{10}$$

For any r, the product pairs in [ ] in (8) are conjugate, since one may be derived from the other by changing the sign of i. Thus if z is real, we have

$$\left.\begin{matrix} ce \\ se \end{matrix}\right\}_{2n+\beta} (z, q) = \left.\begin{matrix} C \\ -S \end{matrix}\right\} \sum_{r=-\infty}^{\infty} (-1)^r A_{2r} \begin{matrix} Re \\ Imag \end{matrix} [J_r(v_1') J_{r+\beta}(v_2')], \tag{11}$$

where the first product pair in (8) is expressed in the form $(x + iy)$.

If the multipliers (9), (10) are used with (6), the respective representations of $Ce_{2n+\beta}(z, q)$ and $Se_{2n+\beta}(z, q)$ are obtained.

To derive the series for ce, se, when $q < 0$, we write $(\frac{1}{2}\pi - z)$ for z in (8), select the real part, redetermine the multipliers, and append $(-1)^n$—see (11) §1. Then we get

$$\left.\begin{matrix} ce \\ se \end{matrix}\right\}_{2n+\beta}(z, -q) = (-1)^n \left.\begin{matrix} C \\ iS \end{matrix}\right\} \sum_{r=-\infty}^{\infty} (-1)^r A_{2r}[I_r(v_1')I_{r+\beta}(v_2') \pm I_{r+\beta}(v_1')I_r(v_2')], \quad (12)$$

where

$$C = ce_{2n+\beta}(0, -q)/2 \sum_{r=-\infty}^{\infty} (-1)^r A_{2r} I_r(k) I_{r+\beta}(k), \quad (13)$$

and

$$S = se'_{2n+\beta}(0, -q)/2k \sum_{r=-\infty}^{\infty} (-1)^r A_{2r}[I_r(k)I'_{r+\beta}(k) - I_{r+\beta}(k)I'_r(k)]. \quad (14)$$

The r.h.s. of (12) are solutions of equation (2) §1.

Writing iz for z in (12) and omitting the multiplier i, yields

$$\left.\begin{matrix} Ce \\ Se \end{matrix}\right\}_{2n+\beta}(z, -q) = (-1)^n \left.\begin{matrix} C \\ S \end{matrix}\right\} \sum_{r=-\infty}^{\infty} (-1)^r A_{2r}[I_r(v_1)I_{r+\beta}(v_2) \pm I_{r+\beta}(v_1)I_r(v_2)], \quad (15)$$

these being solutions of equation (4) §1. The convergence properties of (8), (12), (15) are those stated hitherto.

3. Representation of even solution of (3) §1 in Bessel function series. Substituting $u = 2k \cosh z$, $k = q^{\frac{1}{2}}$ real and > 0, into the equation leads to [4]

$$(u^2 - 4k^2)y'' + uy' + (u^2 - p^2)y = 0, \quad (1)$$

with $p^2 = (a + 2k^2)$. Assume that

$$y = \sum_{r=-\infty}^{\infty} (-1)^r c_{2r} J_{2r+\beta}(u) \quad (2)$$

is a solution of (1), $0 < \beta < 1$. Write $\nu = (2r + \beta)$, insert (2) into (1), and we get

$$\sum_{r=-\infty}^{\infty} (-1)^r c_{2r}[(u^2 - 4k^2)J_\nu'' + uJ_\nu' + (u^2 - p^2)J_\nu] = 0. \quad (3)$$

From Bessel's equation

$$u^2 J_\nu'' + uJ_\nu' + u^2 J_\nu = \nu^2 J_\nu, \quad (4)$$

and by recurrence relations

$$4J_\nu'' = J_{\nu-2} + J_{\nu+2} - 2J_\nu. \quad (5)$$

Substituting (4), (5) into (3), yields

$$\sum_{r=-\infty}^{\infty} (-1)^r c_{2r}[\nu^2 J_r - k^2(J_{r-2} + J_{r+2} - 2J_r) - p^2 J_r] = 0, \tag{6}$$

or

$$\sum_{r=-\infty}^{\infty} (-1)^r c_{2r}[(\nu^2 - a)J_r - k^2(J_{r-2} + J_{r+2})] = 0, \tag{7}$$

since from above $(2k^2 - p^2) = -a$. Equating the coefficient of $J_r(u)$ to zero, $r = -\infty$ to $+\infty$, we obtain

$$[a - (2r + \beta)^2]c_{2r} - k^2(c_{2r+2} + c_{2r-2}) = 0. \tag{8}$$

This is a second order linear difference equation of the type considered in §2. For convergence of (2) above, we need the solution of (8) which tends to zero as $r \to \pm\infty$. Since (8) has the same form as the recurrence relation for the A_{2r} at (7) §1, it follows that $c_{2r} = KA_{2r}$, where K is a constant. Hence

$$\sum_{r=-\infty}^{\infty} (-1)^r A_{2r} J_{2r+\beta}(2k \cosh z) \tag{9}$$

is a solution of (3) §1. It has the following properties:

i. It is an even function of z.

ii. It has a branch point in the u-plane at the origin, and in the z-plane at $z = (2m + 1)\frac{1}{2}\pi i$, $m = -\infty$ to $+\infty$, i.e. on the imaginary axis.

iii. It is multivalued, and if $\beta = p/s$, a rational fraction in its lowest terms, it may be shown to have period $2s\pi i$ in z, $s \geq 2$.

iv. It is absolutely and uniformly convergent if $|\cosh z| > 1$, i.e. if z is not on the imaginary axis, but it is non-uniformly convergent in the neighborhood of $|\cosh z| = 1$, being divergent when $|\cosh z| \leq 1$.

Now $Ce_{2n+\beta}(z, q)$ as defined at (9) §1, is an even function of z, and a solution of the same differential equation as (9) above, while if $\beta = p/s$ it has period $2s\pi i$ in z. Its only singularity in the z-plane is an essential one at infinity. Hence we infer that one solution is a constant multiple of the other, except when z is on the imaginary axis. In this neighborhood, (9) §1 and the corresponding series in Bessel function products, is the analytical continuation of (9) above. Consequently we write

$$Ce_{2n+\beta}(z, q) = C_{2n+\beta} \sum_{r=-\infty}^{\infty} (-1)^r A_{2r} J_{2r+\beta}(2k \cosh z), \tag{10}$$

where $C_{2n+\beta}$ is a constant to be determined. Since the r.h.s. of (10) diverges when $z = 0$, some point at which it is convergent must be chosen to ascertain $C_{2n+\beta}$. Thus taking $z = z_0$, not on the imaginary axis, we find from (10) that

$$C_{2n+\beta} = Ce_{2n+\beta}(z_0, q) \Big/ \sum_{r=-\infty}^{\infty} (-1)^r A_{2r} J_{2r+\beta}(2k \cosh z_0), \tag{11}$$

with

$$Ce_{2n+\beta}(z_0, q) = \sum_{r=-\infty}^{\infty} A_{2r} \cosh (2r + \beta)z_0 . \tag{12}$$

The value of z_0 should be chosen such that the rates of convergence of the series in (11), (12) are of the same order. In this respect, the fact that $J_{-2r+\beta}$ is large for small arguments, should be borne in mind.

When $\beta = 0$, it may be shown that (10) yields the known representation of $Ce_{2n}(z, q)$ in Bessel function series [1, 8].

4. Odd Solution of (3) §1 in Bessel function series. In this case the form of solution at (2) §3 cannot give the desired result, since it is even in z. Accordingly we assume that

$$y = \tanh z\, w(u), \tag{1}$$

where $w(u)$ represents the r.h.s. of (2) §3. The transformed version of (3) §1 is now

$$(u^2 - 4k^2)w'' + uw' + (u^2 - p^2)w + 8k^2 \frac{d}{du}(w/u) = 0. \tag{2}$$

Proceeding as in §3, and using the additional formula

$$4\nu\, d\,[J_\nu(u)/u]/du = J_{\nu-2} - J_{\nu+2}\,, \tag{3}$$

we ultimately obtain the recurrence relation

$$[a - (2r+\beta)^2]\frac{c_{2r}}{2(r+\beta)} - k^2\left[\frac{c_{2r+2}}{(2r+2+\beta)} + \frac{c_{2r-2}}{(2r-2+\beta)}\right] = 0\,. \tag{4}$$

Treating this as a linear difference equation as in §2, if K_1 is a constant, we deduce that $c_{2r}/(2r+\beta) = K_1 A_{2r}$. Hence it follows that

$$\tanh z \sum_{r=-\infty}^{\infty} (-1)^r (2r+\beta) A_{2r} J_{2r+\beta}(2k\cosh z) \tag{5}$$

is a solution of (3) §1, odd in z and linearly independent of (10) §3. It has the properties ii–iv in §3, and we infer that

$$Se_{2n+\beta}(z, q) = S_{2n+\beta} \tanh z \sum_{r=-\infty}^{\infty} (-1)^r (2r+\beta) A_{2r} J_{2r+\beta}(2k\cosh z), \tag{6}$$

where $S_{2n+\beta}$ is a constant. Since series (6) becomes non-uniformly convergent as $z \to 0$, term by term differentiation is invalid in the neighborhood, so $S_{2n+\beta}$ cannot be derived by this process and the substitution $z = 0$. Thus we take $z = z_0$, not on the imaginary axis, and obtain

$$S_{2n+\beta} = Se_{2n+\beta}(z_0, q)/\tanh z_0 \sum_{r=-\infty}^{\infty} (-1)^r (2r+\beta) A_{2r} J_{2r+\beta}(2k\cosh z_0) \tag{7}$$

with

$$Se_{2n+\beta}(z_0, q) = \sum_{r=-\infty}^{\infty} A_{2r} \sinh (2r + \beta)z_0. \tag{8}$$

When $\beta = 1$, it may be shown that (6) gives the known representation of $Se_{2n+1}(z, q)$ in Bessel function series [1, 8].

The solutions $Ce_{2n+\beta}(z, q)$, $Se_{2n+\beta}(z, q)$, even and odd in z, respectively, constitute a fundamental system. Series for functions of order $(2n + 1 + \beta)$ may be derived from those above by writing $(1 + \beta)$ for β, and A_{2r+1} for A_{2r}. The A may be computed using the procedure outlined in reference 7.

Since the series diverge if $|\cosh z| \leqslant 1$, it follows that representations of this type for the functions $ce_{2n+\beta}(z, q)$, $se_{2n+\beta}(z, q)$ do not exist.

5. Solutions of (4) §1 in Bessel function series. If in equation (3) §1 we write $(\frac{1}{2}\pi i + z)$ for z, it transforms to (4) §1. Making this substitution in (9) §3, we derive the solution

$$e^{\frac{1}{2}\beta\pi i} \sum_{r=-\infty}^{\infty} A_{2r} I_{2r+\beta}(2k \sinh z). \tag{1}$$

This form has disadvantages: (a) it is neither odd nor even in z, (b) it converges only if $|\sinh z| > 1$. We may derive an even solution from (9) §3 by putting ik for k, and writing $(-1)^r A_{2r}$ for A_{2r}. Omitting the constant multiplier $e^{\frac{1}{2}\beta\pi i}$, we get the solution

$$Ce_{2n+\beta}(z, -q) = \bar{C}_{2n+\beta} \sum_{r=-\infty}^{\infty} (-1)^r A_{2r} I_{2r+\beta}(2k \cosh z). \tag{2}$$

The r.h.s. of (2) is even in z, while it has the properties ii-iv in §3. Also we obtain

$$\bar{C}_{2n+\beta} = (-1)^n Ce_{2n+\beta}(z_0, -q) / \sum_{r=-\infty}^{\infty} (-1)^r A_{2r} I_{2r+\beta}(2k \cosh z_0), \tag{3}$$

where

$$Ce_{2n+\beta}(z_0, -q) = \sum_{r=-\infty}^{\infty} (-1)^r A_{2r} \cosh (2r + \beta)z_0, \tag{4}$$

the multiplier $(-1)^n$ being present, so that when $\beta = 0$, (2) reduces to the known representation of $Ce_{2n}(z, -q)$ [1, 8]. Similarly we obtain

$$Se_{2n+\beta}(z, -q) = \bar{S}_{2n+\beta} \tanh z \sum_{r=-\infty}^{\infty} (-1)^r (2r + \beta) A_{2r} I_{2r+\beta}(2k \cosh z), \tag{5}$$

where

$$\bar{S}_{2n+\beta} = (-1)^n \sum_{r=-\infty}^{\infty} (-1)^r A_{2r} \sinh (2r + \beta)z_0 \Big/ \sum_{r=-\infty}^{\infty} \cdot (-1)^r (2r + \beta) A_{2r} I_{2r+\beta}(2k \cosh z_0). \tag{6}$$

(5) has properties ii–iv in §3, while in (6) z_0 is neither imaginary nor zero. It may be remarked that (3), (5) may also be derived directly from (4) §1 by the procedure employed in §§ 3, 4, except that in (2) §3 J is replaced by I, and the recurrence and other relationships modified by using $i^{\nu}I_{\nu}(u) = J_{\nu}(iu)$.

6. Asymptotic expansion of $Ce_{2n+\beta}(z, q)$, $Se_{2n+\beta}(z, q)$, z large. If in (3) §1 we write $x = -ike^{z}$, it becomes [3],

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \left(1 + \frac{a}{x^2} + \frac{k^6}{x^4}\right)y = 0. \tag{1}$$

A second transformation is effected by putting $y = we^{z}$, and we get [3, 5],

$$\frac{d^2w}{dx^2} - \left(2 - \frac{1}{x}\right)\frac{dw}{dx} - \left(\frac{1}{x} + \frac{a}{x^2} + \frac{k^6}{x^4}\right)w = 0. \tag{2}$$

To solve (2) in descending powers of x, assume that $w = x^{-\frac{1}{2}}\sum_{r=0}^{\infty}(-1)^r c_r x^{-r}$, with $c_0 = 1$, and substitute this into the equation. Equating coefficients of like powers of x to zero yields [3, 5]:

$$c_1 = \frac{-(4a - 1^2)}{1!\,8}, \qquad c_2 = \frac{(4a - 1^2)(4a - 3^2)}{2!\,8^2} \tag{3}$$

$$c_3 = \frac{-(4a - 1^2)(4a - 3^2)(4a - 5^2)}{3!\,8^3} - \frac{1}{3!}k^4, \tag{4}$$

$$c_4 = \frac{(4a - 1^2)\cdots(4a - 7^2)}{4!\,8^4} + \frac{k^2}{2.4!}(4a - 13). \tag{5}$$

Additional c may be obtained by aid of the recurrence relation ($r \geqslant 2$):

$$2(r + 1)c_{r+1} + [a - (r + \tfrac{1}{2})^2]c_r + k^4 c_{r-2} = 0. \tag{6}$$

Writing $v = ke^{z}$, $x = -iv$, and substituting into $y = we^{-z}$, leads to the solution

$$y = v^{-\frac{1}{2}}e^{i(v+\frac{1}{4}\pi)}(P - iQ), \tag{7}$$

where $P = \sum_{r=0}^{\infty}(-1)^r c_{2r} v^{-2r}$, and $Q = \sum_{r=0}^{\infty}(-1)^r c_{2r+1} v^{-2r-1}$.

Now the real and imaginary parts of (7) are linearly independent solutions, so we have

$$y_1(z) = v^{-\frac{1}{2}}[P\cos(v + \tfrac{1}{4}\pi) + Q\sin(v + \tfrac{1}{4}\pi)], \tag{8}$$

and

$$y_2(z) = v^{-\frac{1}{2}}[P\sin(v + \tfrac{1}{4}\pi) - Q\cos(v + \tfrac{1}{4}\pi)]. \tag{9}$$

Multiplying (9) by $\cos\frac{1}{2}\beta\pi$, (8) by $\sin\frac{1}{2}\beta\pi$, and subtracting the second from the first yields the composite solution

$$y(z) = v^{-\frac{1}{2}}[P\sin(v + \tfrac{1}{4}\pi - \tfrac{1}{2}\beta\pi) - Q\cos(v + \tfrac{1}{4}\pi - \tfrac{1}{2}\beta\pi)]. \tag{10}$$

When $z \to +\infty$, (10) may be written

$$y(z) \sim v^{-\frac{1}{2}} \sin (v + \tfrac{1}{4}\pi - \tfrac{1}{2}\beta\pi). \tag{11}$$

When $z \to +\infty$ in (10) §3, we may use the dominant terms in the asymptotic expansions of the J-Bessel functions, k being finite. Then we obtain

$$Ce_{2n+\beta}(z, q) \sim C_{2n+\beta}(2/\pi v)^{\frac{1}{2}} \sin (v + \tfrac{1}{4}\pi - \tfrac{1}{2}\beta\pi) \sum_{r=-\infty}^{\infty} A_{2r}, \tag{12}$$

$$= C_{2n+\beta}\, ce_{2n+\beta}(0, q)(2/\pi v)^{\frac{1}{2}} \sin (v + \tfrac{1}{4}\pi - \tfrac{1}{2}\beta\pi), \tag{13}$$

which is a constant multiple of (11). Hence by (10), (11), (13) we infer that

$$\begin{aligned} Ce_{2n+\beta}(z, q) \sim C_{2n+\beta}\, ce_{2n+\beta}(0, q)(2/\pi v)^{\frac{1}{2}}[P^{(a)}_{2n+\beta} \sin (v + \tfrac{1}{4}\pi - \tfrac{1}{2}\beta\pi) \\ - Q^{(a)}_{2n+\beta} \cos (v + \tfrac{1}{4}\pi - \tfrac{1}{2}\beta\pi)]. \end{aligned} \tag{14}$$

The superscript (a) and subscript $(2n + \beta)$ signify that in computing the coefficients c_r, $a = a_{2n+\beta}$ is to be used.

When $R(u)$ is large enough, $\cosh u \sim \sinh u$, and since the A_{2r} are common to both series in (9) §1, it follows that $Ce_{2n+\beta}(z, q) \sim Se_{2n+\beta}(z, q)$. Thus both functions have the same asymptotic expansion, it being implied that $R(z)$ is large enough. Accordingly the r.h.s. of (14) is the asymptotic expansion of $Se_{2n+\beta}(z, q)$ as well as that of $Ce_{2n+\beta}(z, q)$. The phase angle range is $-\frac{1}{2}\pi <$ phase $z < \frac{1}{2}\pi$.

When $\beta = 0$ in (14) the asymptotic expansion of $Ce_{2n}(z, q)$ is obtained, while if $\beta = 1$ that of $Se_{2n+1}(z, q)$ results [8].

7. Asymptotic expansion for $Ce_{2n+\beta}(z, -q)$, $Se_{2n+\beta}(z, -q)$, z large. This is derived from (14) § 6 by writing $(\frac{1}{2}\pi i + z)$ for z, multiplying by $(-1)^n$ and discarding the multiplier $e^{\frac{1}{2}\beta\pi i}$. Thus after reduction we obtain

$$\left.\begin{matrix} Ce \\ Se \end{matrix}\right\}_{2n+\beta}(z, -q) \sim \frac{(-1)^n C_{2n+\beta}\, ce_{2n+\beta}(0, q)}{(2\pi v)^{\frac{1}{2}}} \times \\ \times \left[e^{v} \sum_{r=0}^{\infty} c_r v^{-r} - ie^{-(v+\beta\pi)i} \sum_{r=0}^{\infty} (-1)^r c_r v^{-r} \right]. \tag{1}$$

Since the functions are real if z is real, the second member in [ ] is used only when z is complex. The phase angle range is $-\frac{1}{2}\pi <$ phase $z < \frac{1}{2}\pi$.

8. Asymptotic formulae for $Ce_{2n+\beta}(z, q)$, $Se_{2n+\beta}(z, q)$ q real, large and positive. In reference 4. pp. 321–323, the following are shown to be independent approximate solutions of (1), §1, when q is real, large and positive:

$$\begin{matrix} y_1 \\ y_2 \end{matrix}(z) = \frac{e^{\pm 2ks\sin z}\left[\begin{matrix}\cos \\ \sin\end{matrix}(\tfrac{1}{2}z + \tfrac{1}{4}\pi)\right]^{2m+1}}{(\cos z)^{m+1}} \left\{ 1 + \frac{(2m+1) \mp (m^2+m+1)\sin z}{8k\cos^2 z} \right. \\ \left. \mp \frac{[(2m+1)^2 + 1 + 8a_0]}{k} \log_e \tan (\tfrac{1}{2}z + \tfrac{1}{4}\pi) \right\} \tag{1.2}$$

If in (1), (2) we write iz for z, the resulting expressions will be approximate solutions of (3) §1, valid if $-\frac{1}{2}\pi < \text{phase } z < \frac{1}{2}\pi$. This is still true if we multiply (1) by $e^{-\frac{1}{2}\beta\pi i}$, and (2) by $e^{\frac{1}{2}\beta\pi i}$. The result of so doing is

$$\frac{y_1}{y_2}(z) = \frac{e^{\pm i(2k\sinh z - \frac{1}{2}\beta\pi)}[\cosh\frac{1}{2}z \mp i\sinh\frac{1}{2}z]^{2m+1}}{2^{m+\frac{1}{2}}(\cosh z)^{m+1}} \times$$
$$\times\left\{1 + \frac{2m+1) \mp (m^2+m+1)\, i\sinh z}{8k\cosh^2 z} \right. \tag{3,4}$$
$$\left. \mp \frac{[(2m+1)^2+1+8\alpha_0]}{k}\log_e\tan(\tfrac{1}{2}iz + \tfrac{1}{4}\pi)\right\}.$$

Now $Ce_{m+\beta}(z, q)$ is an even function of z, while $Se_{m+\beta}(z, q)$ is an odd one. If $\alpha_0 = -\frac{1}{2}(2m^2 + 2m + 1)$, the coefficient of the logarithmic term vanishes. Then $(y_1 + y_2)$ is even, while $(y_1 - y_2)$ is odd in z. Thus omitting a constant multiplier

$$y_1 \pm y_2 = \frac{1}{(\cosh z)^{\frac{1}{2}}}\left\{\left(1 + \frac{2m+1}{8k\cosh^2 z}\right)\begin{matrix}\cos\\ \sin\end{matrix}\varphi_m \right. \tag{5,6}$$
$$\left. \pm\left[\left(\frac{m^2+m+1}{8k}\right)\frac{\tanh z}{\cosh z}\right]\begin{matrix}\sin\\ \cos\end{matrix}\varphi_m\right\}$$

where $\varphi_m = 2k\sinh z - \frac{1}{2}\beta\pi - (2m+1)\tan^{-1}(\tanh\frac{1}{2}z)$. When $z \to +\infty$ we may write

$$y_1 \pm y_2 = \frac{1}{(\cosh z)^{\frac{1}{2}}}\left\{\begin{matrix}\cos\\ \sin\end{matrix}[2k\sinh(z - \tfrac{1}{2}\beta\pi - (2m+1)\tfrac{1}{4}\pi)]\right\}. \tag{7,8}$$

If in (7) $m = 2n$ and in (8) $m = 2n - 1$, with $2k\cosh z \sim 2k\sinh z \sim v = ke^z$, we obtain in both cases

$$(-1)^n(2k/v)^{\frac{1}{2}}\sin(v + \tfrac{1}{4}\pi - \tfrac{1}{2}\beta\pi). \tag{9}$$

When (9) is multiplied by $(-1)^n C_{2n+\beta}\, ce_{2n+\beta}(0, q)(1/\pi k)^{\frac{1}{2}}$, (13)§ 6 is reproduced. Hence from (5) using this multiplier, we infer that when q is large and positive

$$Ce_{2n+\beta}(z, q) \sim C_{2n+\beta}\, ce_{2n+\beta}(0, q)\left(\frac{2}{\pi v}\right)^{\frac{1}{2}} \times$$
$$\times\left\{\left(1 + \frac{4n+1}{8k\cosh^2 z}\right)\cos\varphi_{2n} + \left[\left(\frac{4n^2+2n+1}{8k}\right)\frac{\tanh z}{\cosh z}\right]\sin\varphi_{2n}\right\} \tag{10}$$

Similarly with $m = 2n - 1$, and remembering that $Ce_{2n+\beta}(z, q) \sim Se_{2n+\beta}(z, q)$ when $R(z)$ is large enough, we find that

$$Se_{2n+\beta}(z, q) \sim C_{2n+\beta}\, ce_{2n+\beta}(0, q)\left(\frac{2}{\pi v}\right)^{\frac{1}{2}} \times$$
$$\times\left\{\left(1 + \frac{4n-1}{8k\cosh^2 z}\right)\sin\varphi_{2n-1} - \left[\left(\frac{4n^2-2n+1}{8k}\right)\frac{\tanh z}{\cosh z}\cos\varphi_{2n-1}\right\}\right. \tag{11}$$

The phase angle range in (10), (11) is $-\frac{1}{2}\pi < \text{phase } z < \frac{1}{2}\pi$.

The asymptotic formulae for q large and negative may be derived from (10), (11) by writing $(\frac{1}{2}\pi i + z)$ for z, multiplying by $(-1)^n$ and discarding the multiplier $e^{\frac{1}{2}\beta\pi i}$. The resulting expression is complex. When z is real, the real part is chosen. The phase angle range is that above.

9. Integral equations for $Ce_{2n+\beta}(z, q)$, $Se_{2n+\beta}(z, q)$. We commence with the formula

$$K_\rho(z) = \int_0^\infty e^{-z\cosh u} \cosh(\rho u)\, du \tag{1}$$

which converges uniformly with respect to z if $R(z) \geqslant x_0$, $x_0 > 0$. Since

$$K_\rho(z) = \tfrac{1}{2}\pi e^{\frac{1}{2}\nu(\rho+1)i}[J_\rho(iz) + iY_\rho(iz)], \tag{2}$$

if we write $(-2ik\cosh z)$ for z and take $R(-2ik\cosh z) \geqslant x_0$, $\rho = (2r + \beta)$, we obtain

$$\begin{aligned} K_\rho(-2ik\cosh z) &= (-1)^r \tfrac{1}{2}\pi e^{\frac{1}{2}\nu(\beta+1)i}[J_\rho(2k\cosh z) + iY_\rho(2k\cosh z)] \\ &= \int_0^\infty e^{2ik\cosh z\cosh u}\cosh(\rho u)\, du, \end{aligned} \tag{4}$$

which converges under the conditions stated above. Then with $\nu = (2n + \beta)$,

$$\begin{aligned} &\tfrac{1}{2}\pi e^{\frac{1}{2}\nu(\beta+1)i}\left[\sum_{r=-\infty}^{\infty}(-1)^r A_{2r}^{(\nu)} J_\rho(2k\cosh z) + i\sum_{r=-\infty}^{\infty}(-1)^r A_{2r}^{(\nu)} Y_\rho(2k\cosh z)\right] \\ &\quad = \sum_{r=-\infty}^{\infty}(-1)^r A_{2r}^{(\nu)} \int_0^\infty e^{2ik\cosh z\cosh u}\cosh(\rho u)\, du \\ &\quad = \int_0^\infty e^{2ik\cosh z\cosh u} Ce_\nu(u, q)\, du, \end{aligned} \tag{5}$$

term by term integration being permissible, since the series are uniformly convergent if $|\cosh z| > 1$, while integral (5) is uniformly convergent with respect to z if $Imag\,(2k\cosh z) > 0$. The first $\sum$ in (5) represents $Ce_\nu(z, q)/C_\nu$, C_ν being the constant multiplier in (11) § 3. If we write $Ye_\nu(z, q)/C_\nu$ for the second $\sum$, we obtain

$$\begin{aligned} &\sin\tfrac{1}{2}\beta\pi\, Ce_\nu(z, q) + \cos\tfrac{1}{2}\beta\pi\, Ye_\nu(z, q) - i[\cos\tfrac{1}{2}\beta\pi\, Ce_\nu(z, q) - \sin\tfrac{1}{2}\beta\pi\, Ye_\nu(z, q)] \\ &\qquad = -(2C_\nu/\pi)\int_0^\infty e^{2ik\cosh z\cosh u} Ce_\nu(u, q)\, du. \end{aligned} \tag{6}$$

Equating real and imaginary parts yields

$$\begin{aligned} &\begin{matrix}\sin\\ \cos\end{matrix}(\tfrac{1}{2}\beta\pi)\, Ce_\nu(z, q) \pm \begin{matrix}\cos\\ \sin\end{matrix}(\tfrac{1}{2}\beta\pi) Ye_\nu(z, q) \\ &\qquad = \mp(2C_\nu/\pi)\int_0^\infty \begin{matrix}\cos\\ \sin\end{matrix}(2k\cosh z\cosh u) Ce_\nu(u, q)\, du. \end{aligned} \tag{7, 8}$$

Then from (7), (8) it follows that

$$Ce_\nu(z, q) = (2C_\nu/\pi) \int_0^\infty \sin(2k \cosh z \cosh u - \tfrac{1}{2}\beta\pi) Ce_\nu(u, q)\, du, \tag{9}$$

and

$$Ye_\nu(z, q) = -(2C_\nu/\pi) \int_0^\infty \cos(2k \cosh z \cosh u - \tfrac{1}{2}\beta\pi) Ce_\nu(u, q)\, du. \tag{10}$$

When $\beta \neq 0$, $Ye_\nu(z, q)$ is *not* a solution of (3) § 1, but if $\beta = 0$ (10) yields the integral relation for the function $Fey_{2n}(z, q)$, while (9) yields that for $Ce_{2n}(z, q)$ [8]. Both functions of order $2n$ are solutions of (3) § 1. Now $Fey_{2n}(z, q)$ is neither odd nor even, being in fact a linear combination of an even solution and an odd one. But $\cos(2k \cosh z \cosh u)$ is even in z, so the infinite integral cannot represent $Fey_{2n}(z, q)$ if, say, z is real < 0. The integral converges if $(2k \cosh z)$ is real but non-zero. Our analysis is based upon k real > 0, so we take z real $\geqslant 0$.

If we commence with the formula [6]

$$\rho K_\rho(z) = z \int_0^\infty e^{-z \cosh u} \sinh u \sinh(\rho u)\, du, \tag{11}$$

$R(z) \geqslant x_0$, $x_0 > 0$, and proceed as above, we obtain

$$Se_\nu(z, q) = -(4/\pi) S_\nu k \sinh z \times \times \int_0^\infty \cos(2k \cosh z \cosh u - \tfrac{1}{2}\beta\pi) \sinh u\, Se_\nu(u, q)\, du, \tag{12}$$

and

$$\hat{Y}e_\nu(z, q) = -(4/\pi) S_\nu k \sinh z \times \times \int_0^\infty \sin(2k \cosh z \cosh u - \tfrac{1}{2}\beta\pi) \sinh u\, Se_\nu(u, q)\, du. \tag{13}$$

The latter is a solution of (3) §1 only when $\beta = 1$, and the integral relation for $Gey_{2n+1}(z, q)$ is reproduced, while (12) yields that for $Se_{2n+1}(z, q)$ [8]. In (12), (13) k, z real > 0.

10. Degeneration of Mathieu functions to Bessel functions of equal order. In (1) § 6 let $k \to 0$, $z \to +\infty$ such that x remains constant. Then we get the equation for the modified Bessel functions $I_\nu(x)$, $K_\nu(x)$, with $\nu = a^{\frac{1}{2}}$. Writing ix for x, the equation for $J_\nu(x)$, $Y_\nu(x)$ results. The condition $z \to +\infty$ entails $h \cosh z \sim r$, the radius vector of a point on the fundamental ellipse, whose interfocal distance is $2h$. Since $2k = k_1 h$, we get $2k \cosh z \sim k_1 r$, and as $h \to 0$, the ellipse tends to a circle of radius r. Applying the above transition formulae to (14) § 6, apart from a constant multiplier, the asymptotic expansion degenerates to that for $J_{2n+\beta}(k_1 r)$ with $\nu = (2n + \beta) = a^{\frac{1}{2}}$. Thus under the above conditions $Ce_{2n+\beta}(z, q)$, $Se_{2n+\beta}(z, q)$ is a constant multiple of $J_{2n+\beta}(k_1 r)$. Simi-

larly (1) § 7 degenerates to a constant multiple of $I_\beta(k_1 r)$. If $n = 0$,* (9), (10) §9 yield the known integrals

$$J_\beta(k_1 r) = \frac{2}{\pi} \int_0^\infty \sin\,(k_1 r \cosh u - \tfrac{1}{2}\beta\pi) \cosh(\beta u)\, du, \tag{1}$$

and

$$Y_\beta(k_1 r) = -\frac{2}{\pi} \int_0^\infty \cos\,(k_1 r \cosh u - \tfrac{1}{2}\beta\pi) \cosh\,(\beta u)\, du, \tag{2}$$

$k_1 r$ real > 0. These integrals have been derived for β real $0 < \beta < 1$, but they are valid for the range $-1 < R(\beta) < 1$ [9].

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* If $n > 0$ the degenerate integrals diverge.

ELECTROMAGNETIC FIELDS IN A PARABOLOIDAL REFLECTOR

By Edmund Pinney

Introduction. In an earlier paper in this journal, [1], the author presented a systematic development of the theory of Laguerre functions with a view to their application to the theory of the paraboloidal reflector. In this paper we will find the electromagnetic fields in an infinite paraboloid when radiating dipoles are placed at its focus in various arrangements.

For continuity and ease of reference, the formula numbering system in this paper is a continuation of that used in [1]. All formula numbers smaller than (6.1) will be found in that paper.

The fields inside the reflector are obtained when it is fed by a radiating dipole at the focus oriented parallel to the axis in §6, by a radiating dipole at the focus oriented perpendicular to the axis in §7, and by a radiating dipole at the focus mounted perpendicular to the axis and backed by a dummy reflector in §8. Since the latter case is of more practical importance, it is developed in more detail than the others.

The parabolic coordinates ξ, η, φ are defined in terms of rectangular coordinates x, y, z in (1.1) Our paraboloid is denoted by $\xi = \xi_0$. The electromagnetic field is assumed to be harmonic of circular frequency ω, the time factor entering proportionally in the fields as $e^{-i\omega t}$. A quantity k is denfied as $k = \omega\sqrt{\mu\epsilon}$, μ and ϵ being the specific magnetic and electric inductive capacities.

6. Paraboloid Fed by Focal Dipole Oriented Parallel to the Axis. In the paraboloidal coordinate system considered in §1, the dipole is at the origin and is oriented parallel to the z-axis. We may denote the paraboloid by $\xi = \xi_0$. In this case the field will be transverse magnetic to the z-axis, and can therefore be described by the z-component o a Hertz vector $\prod = \mathbf{i}_z\psi$, say. From [2; (56), (57), p. 31], the electric and magnetic fields are

$$\mathbf{E} = \nabla(\nabla\cdot\mathbf{\Pi}) + k^2\mathbf{\Pi}, \quad \mathbf{H} = -i\omega\epsilon\nabla\times\mathbf{\Pi}. \tag{6.1}$$

From (1.4),

$$\nabla\cdot(\mathbf{i}_z\psi) = \mathbf{i}_z\cdot\nabla\psi,$$
$$\nabla\psi - \mathbf{i}_\xi\sqrt{\frac{\xi}{\xi+\eta}}\,\psi_\xi + \mathbf{i}_\eta\sqrt{\frac{\eta}{\xi+\eta}}\,\psi_\eta + \mathbf{i}_\varphi\frac{1}{2\sqrt{\xi\eta}}\,\psi_\varphi \tag{6.2}$$

Therefore

$$\nabla\cdot\prod = (-\xi\psi_\xi + \eta\psi_\eta)/(\xi+\eta),$$

so that

$$\left.\begin{aligned} E_\xi &= \sqrt{\frac{\xi}{\xi+\eta}}\left\{\frac{\partial}{\partial\xi}\left(\frac{-\xi\psi_\xi+\eta\psi_\eta}{\xi+\eta}\right) - k^2\psi\right\},\\ E_\eta &= \sqrt{\frac{\eta}{\xi+\eta}}\left\{\frac{\partial}{\partial\eta}\left(\frac{-\xi\psi_\xi+\eta\psi_\eta}{\xi+\eta}\right) + k^2\psi\right\},\\ E_\varphi &= 0, \end{aligned}\right\} \tag{6.3}$$

$$\left.\begin{aligned} H_\xi &= \frac{-i\omega\epsilon}{2\sqrt{\xi(\xi+\eta)}}\,\psi_\varphi, \\ H_\eta &= \frac{-i\omega\epsilon}{2\sqrt{\eta(\xi+\eta)}}\,\psi_\varphi, \\ H_\varphi &= +i\omega\epsilon\,\frac{\sqrt{\xi\eta}}{\xi+\eta}\,(\psi_\xi+\psi_\eta). \end{aligned}\right\} \tag{6.4}$$

Let

$$\psi = \psi^0 + \psi', \tag{6.5}$$

where ψ^0 and ψ' denote the incident and reflected fields, respectively. By (5.25),

$$\left.\begin{aligned} \psi^0 = \frac{1}{r}\,e^{ikr} &= 2\pi k \sum_0^\infty{}_p\, V_p(2ik\xi)S_p(-2ik\eta), \qquad \xi > \eta,[z<0], \\ &= 2\pi k \sum_0^\infty{}_p\, S_p(-2ik\xi)V_p(2ik\eta), \qquad \xi < \eta,[z>0]. \end{aligned}\right\} \tag{6.6}$$

The reflected wave should travel in the $+z$ direction, so from (5.33)

$$\psi' = 2\pi k \sum_0^\infty{}_n\, A_n S_n(2ik\xi)S_n(-2ik\eta), \tag{6.7}$$

where the constants A_n are to be determined. Therefore, for $z < 0$, or $\xi > \eta$,

$$\psi = 2\pi k \sum_0^\infty{}_n\, [V_n(2ik\xi) + A_n S_n(2ik\xi)]S_n(-2ik\eta). \tag{6.8}$$

By (5.29),

$$\begin{aligned} -\xi\psi_\xi + \eta\psi_\eta &= ik(\xi+\eta)\psi + 2\pi k \sum_0^\infty{}_n [D_{n-1}(-2ik\eta, 2ik\xi) + A_n C_{n-1}(-2ik\eta, 2ik\xi)] \\ &= ik(\xi+\eta)\psi - 4\pi ik^3(\xi+\eta) \\ &\qquad \cdot \left[\sum_0^\infty{}_n \sum_n^\infty{}_p\, S_p(-2ik\eta)V_p(2ik\xi) - \sum_0^\infty{}_n A_n \sum_0^{n-1}{}_p\, S_p(-2ik\eta)S_p(2ik\xi)\right] \\ &= 2\pi ik^2(\xi+\eta) \sum_0^\infty{}_n\, [V_n(2ik\xi) + A_n S_n(2ik\xi)]S_n(-2ik\eta) - 4\pi ik^3(\xi+\eta) \\ &\qquad \cdot \sum_0^\infty{}_n \left[(n+1)V_n(2ik\xi) - \left(\sum_{n+1}^\infty{}_p\, A_p\right) S_n(2ik\xi)\right] S_n(-2ik\eta), \end{aligned}$$

so that

$$\begin{aligned} &(\eta\psi_\eta - \xi\psi_\xi)/(\xi+\eta) \\ &\quad = 2\pi ik^2 \sum_0^\infty{}_n \left[-(2n+1)V_n(2ik\xi) + \left(A_n + 2\sum_{n+1}^\infty{}_p\, A_p\right)S_n(2ik\xi)\right]S_n(-2ik\eta). \end{aligned}$$

From (5.8), (5.25),

$$d[S_r^m(z)]/dz = \tfrac{1}{2}(\mu z^{-1} - z^{-1})S_r^{m+1}(z) - \tfrac{1}{2}(\mu z^{-1} + z^{-1})S_{r-1}^{m+1}(z), \tag{6.9}$$

with a similar expression for $V_r^n(z)$. Therefore

$$\frac{\partial}{\partial\xi}\left(\frac{-\xi\psi_\xi+\eta\psi_\eta}{\xi+\eta}\right)=-\frac{2\pi k^3}{\sqrt{2ik\xi}}\sum_0^\infty{}_n\Big\{(2n+1)[V_n^1(2ik\xi)+V_{n-1}^1(2ik\xi)]$$
$$-\Big(A_n+2\sum_{n+1}^\infty{}_p A_p\Big)[S_n^1(2ik\xi)+S_{n-1}^1(2ik\xi)]\Big\}S_n(-2ik\eta),$$

$$\frac{\partial}{\partial\eta}\left(\frac{-\xi\psi_\xi+\eta\psi_\eta}{\xi+\eta}\right)=\frac{2\pi k^3}{\sqrt{-2ik\eta}}\sum_0^\infty{}_n\Big\{(2n+1)V_n(2ik\xi)$$
$$-(A_n+2\sum_{n+1}^\infty{}_p A_p)S_n(2ik\xi)\Big\}[S_n^1(-2ik\eta)+S_{n-1}^1(-2ik\eta)].$$

From (6.8) and (5.8),

$$k^2\psi=\frac{2\pi k^3}{\sqrt{-2ik\eta}}\sum_0^\infty{}_n\{V_n(2ik\xi)+A_nS_n(2ik\xi)\}[S_n^1(-2ik\eta)-S_{n-1}^1(-2ik\eta)]$$
$$=\frac{2\pi k^3}{\sqrt{2ik\xi}}\sum_0^\infty{}_n\{V_n^1(2ik\xi)-V_{n-1}^1(2ik\xi)+A_n[S_n^1(2ik\xi)-S_{n-1}^1(2ik\xi)]\}S_n(-2ik\eta)$$

Therefore, if

$$B_n=\sum_n^\infty{}_p A_p \tag{6.10}$$

$$E_\xi=-\frac{4\pi k^3}{\sqrt{2ik(\xi+\eta)}}\sum_0^\infty{}_n\{(n+1)V_n^1(2ik\xi)+nV_{n-1}^1(2ik\xi)$$
$$-B_{n+1}S_n^1(2ik\xi)-B_nS_{n-1}^1(2ik\xi)\}S_n(-2ik\eta), \tag{6.11}$$

$$E_\eta=\frac{4\pi k^3}{\sqrt{-2ik(\xi+\eta)}}\sum_0^\infty{}_n\{(n+1)]V_n(2ik\xi)+V_{n+1}(2ik\xi)]$$
$$-B_{n+1}[S_n(2ik\xi)+S_{n+1}(2ik\xi)]\}S_n^1(-2ik\eta) \tag{6.12}$$

By (5.31), (5.27), (5.28)

$$\psi_\xi+\psi_\eta=-\frac{2\pi ik}{\sqrt{\xi\eta}}\sum_0^\infty{}_n[D_{n-1}^1(-2ik\eta,\,2ik\xi)+A_nC_{n-1}^1(-2ik\eta,\,2ik\xi)]$$
$$=-4\pi k^2\frac{\xi+\eta}{\sqrt{\xi\eta}}\Big\{\sum_0^\infty{}_n\sum_n^\infty{}_p\frac{1}{p+1}S_p^1(-2ik\eta)V_p^1(2ik\xi)$$
$$-\sum_0^\infty{}_nA_n\sum_0^{n-1}{}_p\frac{1}{p+1}S_p^1(-2ik\eta)S_p^1(2ik\xi)\Big\}.$$

Therefore

$$\sqrt{\xi\eta}(\psi_\xi+\psi_\eta)/(\xi+\eta)=$$
$$-4\pi k^2\sum_0^\infty{}_n[V_n^1(2ik\xi)-B_{n+1}S_n^1(2ik\xi)/(n+1)]S_n^1(-2ik\eta), \tag{6.13}$$

so by (6.4),

$$H_\varphi = -4\pi i k^2 \omega\epsilon \sum_0^\infty{}_n [V_n^1(2ik\xi) - B_{n+1} S_n^1(2ik\xi)/(n+1)] S_n^1(-2ik\eta). \quad (6.14)$$

Now $E_\eta = 0$ when $\xi = \xi_0$, so by (6.12),

$$B_n = n[V_n(2ik\xi_0) + V_{n-1}(2ik\xi_0)]/[S_n(2ik\xi_0) + S_{n-1}(2ik\xi_0)] \quad (6.15)$$

This expression may be substituted into (6.11), (6.12), (6.14), giving E_ξ, E_η, H_φ. The components E_φ, H_ξ, H_η are all zero.

7. **Paraboloid Fed by Focal Dipole Oriented Perpendicular to the Axis.** In the paraboloidal coordinate system of §1, the dipole is at the origin and is oriented parallel to the x-axis, say. We denote the paraboloid by $\xi = \xi_0$. In this case the incident field will be transverse magnetic to the x-axis.

We will now compute the fields due to the dipole alone. From [2; (56), (57), p. 31], the electric and magnetic fields are given by

$$\mathbf{E}^\circ = \nabla(\nabla\cdot\mathbf{\Pi}) + k^2\mathbf{\Pi}, \mathbf{H}^\circ = -i\omega\epsilon\nabla\times\mathbf{\Pi}, \quad (7.1)$$

$$\mathbf{\Pi} = \psi \mathbf{i}_z, \quad (7.2)$$

ψ being independent of φ.

From (1.4),

$$\left.\begin{aligned} \mathbf{i}_x &= \sqrt{\frac{\eta}{\xi+\eta}}\cos\varphi\,\mathbf{i}_\xi + \sqrt{\frac{\xi}{\xi+\eta}}\cos\varphi\,\mathbf{i}_\eta - \sin\varphi\,\mathbf{i}_\varphi, \\ \mathbf{i}_y &= \sqrt{\frac{\eta}{\xi+\eta}}\sin\varphi\,\mathbf{i}_\xi + \sqrt{\frac{\xi}{\xi+\eta}}\sin\varphi\,\mathbf{i}_\eta + \cos\varphi\,\mathbf{i}_\varphi, \\ \mathbf{i}_z &= -\sqrt{\frac{\xi}{\xi+\eta}}\,\mathbf{i}_\xi + \sqrt{\frac{\eta}{\xi+\eta}}\,\mathbf{i}_\eta. \end{aligned}\right\} \quad (7.3)$$

From (6.2),

$$\nabla = \sqrt{\frac{\xi}{\xi+\eta}}\,\mathbf{i}_\xi\frac{\partial}{\partial\xi} + \sqrt{\frac{\eta}{\xi+\eta}}\,\mathbf{i}_\eta\frac{\partial}{\partial\eta} + \frac{1}{2\sqrt{\xi\eta}}\,\mathbf{i}_\varphi\frac{\partial}{\partial\varphi}, \quad (7.4)$$

so

$$\nabla\cdot\mathbf{\Pi} = \mathbf{i}_z\cdot\nabla\psi = p\cos\varphi,$$

where

$$p = \sqrt{\xi\eta}(\psi_\xi + \psi_\eta)/2(\xi+\eta). \quad (7.5)$$

Therefore

$$E_\xi^0 = \frac{1}{\sqrt{\xi(\xi+\eta)}}(2\xi p_\xi + k^2\sqrt{\xi\eta}\,\psi)\cos\varphi,$$

$$E_\eta^0 = \frac{1}{\sqrt{\eta(\xi+\eta)}}(2\eta p_\eta + k^2\sqrt{\xi\eta}\,\psi)\cos\varphi,$$

$$E_\varphi^0 = -\frac{1}{\sqrt{\xi\eta}}(p + k^2\sqrt{\xi\eta}\,\psi)\sin\varphi.$$

From (7.3),

$$\left.\begin{aligned} E_x^0 &= \frac{\sqrt{\xi\eta}}{\xi+\eta}(p_\xi + p_\eta) + \frac{1}{2\sqrt{\xi\eta}}p + k^2\psi \\ &\qquad + \left\{\frac{\sqrt{\xi\eta}}{\xi+\eta}(p_\xi + p_\eta) - \frac{1}{2\sqrt{\xi\eta}}p\right\}\cos 2\varphi, \\ E_y^0 &= \left\{\frac{\sqrt{\xi\eta}}{\xi+\eta}(p_\xi + p_\eta) - \frac{1}{2\sqrt{\xi\eta}}p\right\}\sin 2\varphi, \\ E_z^0 &= \frac{2}{\xi+\eta}(-\xi p_\xi + \eta p_\eta)\cos\varphi. \end{aligned}\right\} \tag{7.6}$$

From (6.6),

$$\psi = 2\pi k \sum_0^\infty{}_n V_n(2ik\xi)S_n(-2ik\eta), \qquad \xi > \eta. \tag{7.7}$$

From (6.13), (7.5),

$$p = -2\pi k^2 \sum_0^\infty{}_n V_n^1(2ik\xi)S_n^1(-2ik\eta).$$

From (5.30), (5.27), (5.31),

$$\frac{\sqrt{\xi\eta}}{\xi+\eta}(p_\xi + p_\eta) + \frac{1}{2\sqrt{\xi\eta}}p = 2\pi k^2 \sum_0^\infty{}_n n(n+1)V_n(2ik\xi)S_n(-2ik\eta),$$

$$\frac{\sqrt{\xi\eta}}{\xi+\eta}(p_\xi + p_\eta) - \frac{1}{2\sqrt{\xi\eta}}p = 2\pi k^2 \sum_0^\infty{}_n V_n^2(2ik\xi)S_n^2(-2ik\eta).$$

From (5.27), (5.29),

$$2(-\xi p_\xi + \eta p_\eta)/(\xi+\eta) = 4\pi i k^2 \sum_0^\infty{}_n (n+1)V_n^1(2ik\xi)S_n^1(-2ik\eta).$$

In view of these results, and of the fact that the reflected wave will travel in the $+z$ direction,

$$E_x = 2\pi k^2\{f(\xi,\eta) + h(\xi,\eta)\cos 2\varphi\},\quad E_y = 2\pi k^2 h(\xi,\eta)\sin 2\varphi,$$
$$E_z = 2\pi k^2 g(\xi,\eta)\cos\varphi, \tag{7.8}$$

where

$$\left.\begin{aligned} f(\xi,\eta) &= \sum_0^\infty{}_n [(n^2+n+1)V_n(2ik\xi) - A_n S_n(2ik\xi)]S_n(-2ik\eta), \\ g(\xi,\eta) &= 2i\sum_0^\infty{}_n [(n+1)V_n^1(2ik\xi) - B_n(n+1)^{-1}S_n^1(2ik\xi)]S_n^1(-2ik\eta), \\ h(\xi,\eta) &= \sum_0^\infty [V_n^2(2ik\xi) - C_n(n+1)^{-1}(n+2)^{-1}S_n^2(2ik\xi)]S_n^2(-2ik\eta). \end{aligned}\right\} \tag{7.9}$$

Now $\nabla \cdot \mathbf{E} = 0$, so since

$$\frac{\partial}{\partial x} = \frac{\sqrt{\xi\eta}}{\xi+\eta}\cos\varphi\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right) - \frac{\sin\varphi}{2\sqrt{\xi\eta}}\frac{\partial}{\partial\varphi},$$

$$\frac{\partial}{\partial y} = \frac{\sqrt{\xi\eta}}{\xi+\eta}\sin\varphi\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right) + \frac{\cos\varphi}{2\sqrt{\xi\eta}}\frac{\partial}{\partial\varphi},$$

$$\frac{\partial}{\partial z} = \frac{1}{\xi+\eta}\left(-\xi\frac{\partial}{\partial\xi}+\eta\frac{\partial}{\partial\eta}\right),$$

we get

$$\frac{\sqrt{\xi\eta}}{\xi+\eta}(f_\xi+f_\eta)+\frac{1}{\xi+\eta}(-\xi g_\xi+\eta g_\eta)+\frac{\sqrt{\xi\eta}}{\xi+\eta}(h_\xi+h_\eta)+\frac{1}{\sqrt{\xi\eta}}h=0. \quad (7.10)$$

By (5.31), (5.27), (5.28),

$$\frac{\sqrt{\xi\eta}}{\xi+\eta}(f_\xi+f_\eta) = -2k\sum_0^\infty{}_n\left\{\tfrac{1}{3}(n^2+2n+3)V_n^1(2ik\xi) + \frac{1}{n+1}\left(\sum_{n+1}^\infty{}_p A_p\right)S_n^1(2ik\xi)\right\}S_n^1(-2ik\eta).$$

By (5.29), (5.27), (5.28),

$$\frac{1}{\xi+\eta}(-\xi g_\xi+\eta g_\eta) = 2k\sum_0^\infty{}_n\left\{\tfrac{1}{3}(2n^2+4n+3)V_n^1(2ik\xi) + \frac{1}{n+1}\left(B_n+2\sum_{n+1}^\infty{}_p B_p\right)S_n^1(2ik\xi)\right\}S_n^1(-2ik\eta).$$

By (5.30), (5.27), (5.28),

$$\frac{\sqrt{\xi\eta}}{\xi+\eta}(h_\xi+h_\eta)+\frac{1}{\sqrt{\xi\eta}}h = -2k\sum_0^\infty{}_n\left\{\tfrac{1}{3}n(n+2)V_n^1(2ik\xi) + \frac{1}{n+1}\left(\sum_n^\infty{}_p C_p\right)S_n^1(2ik\xi)\right\}S_n^1(-2ik\eta).$$

Substituting these into (7.10),

$$\sum_{n+1}^\infty{}_p A_p - B_n - 2\sum_{n+1}^\infty{}_p B_p + \sum_n^\infty{}_p C_p = 0.$$

Subtracting,

$$A_n - B_n - B_{n-1} + C_{n-1} = 0. \quad (7.11)$$

From (7.3), (7.8),

$$\left.\begin{aligned} E_\xi &= 2\pi k^3\sqrt{\eta/(\xi+\eta)}\,[f(\xi,\eta)+h(\xi,\eta)-\sqrt{\xi/\eta}\,g(\xi,\eta)]\cos\varphi,\\ E_\eta &= 2\pi k^3\sqrt{\xi/(\xi+\eta)}\,[f(\xi,\eta)+h(\xi,\eta)+\sqrt{\eta/\xi}\,g(\xi,\eta)]\cos\varphi,\\ E_\varphi &= 2\pi k^3[-f(\xi,\eta)+h(\xi,\eta)]\sin\varphi. \end{aligned}\right\} \quad (7.12)$$

Now E_η and E_φ vanish when $\xi = \xi_0$, so

$$f(\xi_0, \eta) - h(\xi_0, \eta) = 0, \tag{7.13}$$

$$2f(\xi_0, \eta) + \sqrt{\eta/\xi_0}\, g(\xi_0, \eta) = 0 \tag{7.14}$$

Hereafter, for brevity we will write the functions of $2ik\xi_0$ without their arguments. That is,

$$V_n^m(2ik\xi_0) = V_n^m, \; S_n^m(2ik\xi_0) = S_n^m, \text{ etc.}$$

From (7.9), making use of(5.8) and (5.5),

$$\sqrt{-2ik\eta}\, f(\xi_0, \eta) = \sum_0^\infty{}' [(n^2 + n + 1)V_n - (n^2 + 3n + 3)V_{n+1} + A_{n+1}S_{n+1} - A_nS_n]S_n^1(-2ik\eta).$$

$$\sqrt{-2ik\eta}\, h(\xi_0, \eta) = \sum_0^\infty{}' [-nV_{n-1}^2 + (n + 2)V_n^2 + (C_{n-1}S_{n-1}^2 - C_nS_n^2)/(n + 1)]S_n^1(-2ik\eta).$$

From (7.13), therefore,

$$\left.\begin{aligned} A_{n+1}S_{n+1} - A_nS_n + (C_nS_n^2 - C_{n-1}S_{n-1}^2)/(n + 1) &= (n^2 + 3n + 3)V_{n+1} \\ &\quad -(n^2 + n + 1)V_n + (n + 2)V_n^2 - nV_{n-1}^2 \end{aligned}\right\} \tag{7.15}$$

From (7.9), (5.5),

$$\sqrt{-2ik\eta}\, g(\xi_0, \eta) = 2i \sum_0^\infty{}' [(n + 1)^2 V_n^1 - n^2 V_{n-1}^1 - B_nS_n^1 + B_{n-1}S_{n-1}^1]S_n(-2ik\eta).$$

From (7.9) and (5.8),

$$-A_n\sqrt{2ik\xi_0}S_n + B_nS_n^1 - B_{n-1}S_{n-1}^1 = nV_n^1 + (n + 1)V_{n-1}^1 \tag{7.16}$$

Let

$$\theta_n = nV_n^1 + (n + 1)V_{n-1}^1. \tag{7.17}$$

Then by (7.11), (7.16),

$$\left.\begin{aligned} A_n\sqrt{2ik\xi_0}S_n &= B_nS_n^1 - B_{n-1}S_{n-1}^1 - \theta_n, \; C_{n-1}\sqrt{2ik\xi_0}S_n = -B_nS_{n-1}^1 \\ &\quad + B_{n-1}S_n^1 + \theta_n. \end{aligned}\right\} \tag{7.18}$$

By (5.5), (5.8), (7.15),

$$\left.\begin{aligned} \sqrt{2ik\xi_0}(A_{n+1}S_{n+1} - A_nS_n) + \sqrt{2ik\xi_0}(C_nS_n^2 - C_{n+1}S_{n-1}^2)/(n + 1) \\ = V_{n+1}^1 + V_{n-1}^1 \end{aligned}\right\} \tag{7.19}$$

Write

$$\left.\begin{aligned} M_n(z) &= \{n[S_n^1(z)]^2 - (n + 1)[S_{n-1}^1(z)]^2\}/S_n(z), \\ N_n(z) &= \{nS_n^1(z)V_n^1(z) - (n + 1)S_{n-1}^1(z)V_{n-1}^1(z)\}/S_n(z). \end{aligned}\right\} \tag{7.20}$$

From (5.5), (5.8),

$$\left.\begin{aligned}\sqrt{2ik\xi_0}[nV_nS^1_n - V^1_{n-1}S^2_{n-1}] = \sqrt{2ik\xi_0}[(n+1)V_nS^1_{n-1} - V^1_nS^2_{n-1}] = S_nN_n\,,\\ \sqrt{2ik\xi_0}[nS_nS^1_n - S^1_{n-1}S^2_{n-1}] = \sqrt{2ik\xi_0}[(n+1)S_nS^1_{n-1} - S^1_nS^2_{n-1}] = S_nM_n\,.\end{aligned}\right\} \quad (7.21)$$

Substituting (7.18) into (7.19),

$$B_{n+1}[(n+1)S^1_{n+1} - S^1_nS^2_n/S_{n+1}] + B_n[-2(n+1)S^1_n + S^1_{n+1}S^2_n/S_{n+1} + S^1_{n-1}S^2_{n-1}/S_n] + B_{n-1}[(n+1)S^1_{n-1} - S^1_nS^2_{n-1}/S_n] = (n+1)[V^1_{n+1} + V^1_{n-1}] + \theta_{n+1}[n+1-S^2_n/S_{n+1}] - \theta_n[n+1-S^2_{n-1}/S_n].$$

From (7.21),

$$B_{n+1}M_{n+1} - B_n(M_{n+1} + M_n) + B_{n-1}M_n = [(n+1)[V^1_{n+1} + V^1_{n-1}) - \theta_n]\sqrt{2ik\xi_0} + [\theta_{n+1}(n+1-S^2_n/S_{n+1}) - \theta_n(n - S^2_{n-1}/S_n)]\sqrt{2ik\xi_0}.$$

By (7.17),

$$(n+1)(V^1_{n+1} + V^1_{n-1}) - \theta_n = (n+1)V^1_{n+1} - nV^1_n\,.$$

Therefore

$$(B_{n+1} - B_n)M_{n+1} - (n+1)V^1_{n+1}\sqrt{2ik\xi_0} - \theta_{n+1}(n+1-S^2_n/S_{n+1})\sqrt{2ik\xi_0} = (B_n - B_{n-1})M_n - nV^1_n\sqrt{2ik\xi_0} - \theta_n(n - S^2_{n-1}/S_n)\sqrt{2ik\xi_0}.$$

From this it follows that

$$(B_n - B_{n-1})M_n = nV^1_n\sqrt{2ik\xi_0} + \theta_n(n - S^2_{n-1}/S_n)\sqrt{2ik\xi_0} + F$$

where F is independent of n.
From (7.17), (7.21), (5.8), (5.35),

$$(B_n - B_{n-1})M_n - (2n+1)N_n - (2i/\pi)n^2(n+1)/S_n + F.$$

Therefore

$$B_n = B + \sum_0^n{}_p\,(2p+1)N_p/M_p - 2i/\pi\sum_0^n{}_p\,p^2(p+1)/(S_pM_p) + F\sum_0^n{}_p\,1/M_p.$$

Now $\sum_0^\infty{}_p\,B_p$ must converge, so $B_n \to 0$ as $n \to \infty$. Therefore

$$B_n = -\sum_{n+1}^\infty{}_p\,(2p+1)N_p/M_p + (2i/\pi)\sum_{n+1}^\infty{}_p\,p^2(p+1)/(S_pM_p) - F\sum_{n+1}^\infty{}_p\,1/M_p$$

From (7.9) $\sum_0^\infty{}_n\,B_n(n+1)^{-1}S^1_n(2ik\xi)S^1_n(-2ik\eta)$ must converge, so by (5.23), (5.24), $F = 0$, and

$$B_n = -\sum_{n+1}^\infty{}_p\,(2p+1)\frac{N_p(2ik\xi_0)}{M_p(2ik\xi_0)} + \frac{2i}{\pi}\sum_{n+1}^\infty{}_p\,\frac{p^2(p+1)}{S_p(2ik\xi_0)M_p(2ik\xi_0)} \quad (7.22)$$

A_n and C_n may be computed from (7.18), (5.8), (7.11). We get

$$A_n = B_n - n\frac{N_n(2ik\xi_0)}{M_n(2ik\xi_0)} + \frac{i}{\pi}\frac{n^2(n+1)}{S_n(2ik\xi_0)M_n(2ik\xi_0)}, \tag{7.23}$$

$$C_{n-1} = B_{n-1} + n\frac{N_n(2ik\xi_0)}{M_n(2ik\xi_0)} - \frac{i}{\pi}\frac{n^2(n+1)}{S_n(2ik\xi_0)M_n(2ik\xi_0)}. \tag{7.24}$$

8. Paraboloid with Focal Dipole which is Oriented Perpendicular to the Axis and which is Backed by a Dummy Reflector. In actual practice the primary radiating dipole is oriented perpendicular to the axis of the paraboloid, and a "dummy reflector" or secondary dipole is placed in front of it on the axis in such a manner as to cause the major part of the primary radiation to go towards the apex of the paraboloid. In this section the results of the last section will be modified in such a manner as to take into account the effect of the added dummy reflector.

In §7, the x-component of the Hertz vector of the incident radiation was

$$\psi^0 = e^{ikr}/r.$$

Now

$$\partial\psi^0/\partial z = (ik/r)[1 + /i/kr)]e^{ikr}z/r$$

Therefore

$$\psi^0 - (1/ik)\partial\psi^0/\partial z = (1/r)[1 - (z/r) - (iz/kr^3)]e^{ikr}.$$

For r large, we see that this quantity is minimum on the positive z-axis, and maximum on the negative z-axis. Therefore, in the coordinate system of §7, $\psi^0 - (1/ik)\partial\psi^0/\partial z)$ will make an appropriate x-component of the incident Hertz vector for the present problem. We may solve the present problem by differentiating the incident fields of §7 with respect to $-ikz$, adding the result to the incident fields, and proceeding as in §7.

The incident fields are obtained from (7.8), (7.9) with the aid of

$$\partial/\partial z = (-\xi\partial/\partial\xi + \eta\partial/\partial\eta)/(\xi + \eta)$$

and (5.29), (5.27). When they have been computed, the total fields may be written in the form

$$E_x = \tfrac{4}{3}\pi k^3[F(\xi, \eta) + H(\xi, \eta)\cos 2\varphi],\ E_y = \tfrac{4}{3}\pi k^3 H(\xi, \eta)\sin 2\varphi,$$
$$E_z = \tfrac{4}{3}\pi k^3 G(\xi, \eta)\cos\varphi, \tag{8.1}$$

where

$$\left.\begin{aligned} F(\xi, \eta) &= \sum_0^\infty{}_n [(n+1)(n^2+2n+3)V_n(2ik\xi) - A_nS_n(2ik\xi)]S_n(-2ik\eta), \\ G(\xi, \eta) &= i\sum_0^\infty{}_n [(n+2)(2n+3)V_n^1(2ik\xi) \\ &\qquad - 2B_n(n+1)^{-1}S_n^1(2ik\xi)]S_n^1(-2ik\eta), \\ H(\xi, \eta) &= \sum_0^\infty{}_n [(n+3)V_n^2(2ik\xi) \\ &\qquad - C_n(n+1)^{-1}(n+2)^{-1}S_n^2(2ik\xi)[S_n^2(-2ik\eta). \end{aligned}\right\} \tag{8.2}$$

The reflected field here is formally the same as that in (7.9), although the constants A_n, B_n, C_n are different. The condition $\nabla \cdot \mathbf{E} = 0$ will therefore give the same relation between these constants. As in (7.11)

$$A_n - B_n - B_{n-1} + C_{n-1} = 0. \tag{8.3}$$

As in (7.12),

$$\left.\begin{aligned} E_\xi &= \tfrac{1}{2}\pi k^3\sqrt{\eta/(\xi+\eta)}[F(\xi,\eta) + H(\xi,\eta) - \sqrt{\xi/\eta}G(\xi,\eta)]\cos\varphi, \\ E_\eta &= \tfrac{1}{2}\pi k^3\sqrt{\xi/(\xi+\eta)}[F(\xi,\eta) + H(\xi,\eta) + \sqrt{\eta/\xi}G(\xi,\eta)]\cos\varphi, \\ E_\varphi &= \tfrac{1}{2}\pi k^3[-F(\xi,\eta) + H(\xi,\eta)]\sin\varphi. \end{aligned}\right\} \tag{8.4}$$

Comparing (7.9) with (8.2), it is readily seen that (7.15) now becomes

$$\left.\begin{aligned} &A_{n+1}S_{n+1} - A_nS_n + (C_nS_n^2 - C_{n-1}S_{n-1}^2)/(n+1) \\ &\quad = (n+2)(n^2+4n+6)V_{n+1} - (n+1)(n^2+2n+3)V_n \\ &\qquad + (n+2)(n+3)V_n^2 - n(n+2)V_{n-1}^2, \end{aligned}\right\} \tag{8.5}$$

and (7.16) becomes

$$-A_n\sqrt{2ik\xi_0}S_n + B_nS_n^1 - B_{n-1}S_{n-1}^1 = \tfrac{1}{2}n(n+1)V_n^1 + \tfrac{1}{2}(n+1)(n+2)V_{n-1}^1 \tag{8.6}$$

Suppose we let

$$\theta_n = \tfrac{1}{2}n(n+1)V_n^1 + \tfrac{1}{2}(n+1)(n+2)V_{n-1}^1 \tag{8.7}$$

Then, as in (7.18),

$$A_n\sqrt{2ik\xi_0}S_n = B_nS_n^1 - B_{n-1}S_{n-1}^1 - \theta_n,\ C_{n-1}\sqrt{2ik\xi_0}S_n = -B_nS_{n-1}^1 + B_{n-1}S_n^1 + \theta_n. \tag{8.8}$$

By (5.5), (5.8), (8.5),

$$\left.\begin{aligned} &\sqrt{2ik\xi_0}(A_{n+1}S_{n+1} - A_nS_n) + \sqrt{2ik\xi_0}(C_nS_n^2 - C_{n-1}S_{n-1}^2)/(n+1) = \\ &\qquad = 3[(n+2)V_{n+1}^1 - (n+1)V_n^1 + (n+1)V_{n-1}^1] \end{aligned}\right\} \tag{8.9}$$

Then, as in §7, below (7.21),

$$B_{n+1}M_{n+1} - B_n(M_{n+1} + M_n) + B_{n-1}M_n = \{3(n+1)[(n+2)V_{n+1}^1 - (n+1)V_n^1 + (n+1)V_{n-1}^1] - \theta_n\}\sqrt{2ik\xi_0} + \{\theta_{n+1}(n+1-S_n^2/S_{n+1}) - \theta_n(n - S_{n-1}^2/S_n)\}\sqrt{2ik\xi_0}$$

By (8.7),

$$3(n+1)[(n+2)V_{n+1}^1 - (n+1)V_n^1 + (n+1)V_{n-1}^1] - \theta_n = \tfrac{1}{2}(n+1)(n-2)(2V_{n+1}^1 - V_n^1) - \tfrac{1}{2}n(n+1)(2V_n^1 - V_{n-1}^1).$$

Therefore

$$(B_{n+1} - B_n)M_{n+1} - \tfrac{1}{2}(n+1)(n+2)(2V_{n+1}^1 - V_n^1)\sqrt{2ik\xi_0} - \theta_{n+1}(n+1-S_n^2/S_{n+1})\sqrt{2ik\xi_0} = (B_n - B_{n-1})M_n - \tfrac{1}{2}n(n+1)(2V_n^1 - V_{n-1}^1)\sqrt{2ik\xi_0} - \theta_n(n - S_{n-1}^2/S_n)\sqrt{2ik\xi_0}$$

From this it follows that

$$(B_n - B_{n-1})M_n = \tfrac{1}{2}n(n+1)(2V_n^1 - V_{n-1}^1)\sqrt{2ik\xi_0} + \theta_n(n - S_{n-1}^2/S_n)\sqrt{2ik\xi_0} + f$$

where f is independent of n.

By (8.7), (7.21), (5.8),

$$(B_n - B_{n-1})M_n = 3(n+1)^2N_n + \tfrac{3}{2}n(n+1)(2n+3)(S_n^1V_{n-1}^1 - S_{n-1}^{11}V_n^1)/S_n + f.$$

By (5.35),

$$B_n - B_{n-1} = 3(n+1)^2N_n/M_n - \frac{3i}{2\pi}n^2(n+1)(2n+3)/(S_nM_n) + f.$$

As in (7.22) it follows that $f = 0$, and

$$B_n = -3\sum_{n+1}^{\infty}{}_p\,(p+1)^2\frac{N_p(2ik\xi_0)}{M_p(2ik\xi_0)} + \frac{3i}{2\pi}\sum_{n+1}^{\infty}{}_p\,\frac{p^2(p+1)(2p+3)}{S_p(2ik\xi_0)M_p(2ik\xi_0)}. \tag{8.10}$$

From (8.8), (7.20), (5.8), (8.3),

$$A_n = B_n - \tfrac{1}{2}n(n+1)\frac{N_n(2ik\xi_0)}{M_n(2ik\xi_0)} + \frac{3i}{2\pi}\frac{n^2(n+1)(n+2)}{S_n(2ik\xi_0)M_n(2ik\xi_0)}, \tag{8.11}$$

$$C_{n-1} = B_{n-1} + \tfrac{1}{2}n(n+1)\frac{N_n(2ik\xi_0)}{M_n(2ik\xi_0)} - \frac{3i}{2\pi}\frac{n^2(n+1)(n+2)}{S_n(2ik\xi_0)M_n(2ik\xi_0)}. \tag{8.12}$$

To get the magnetic field, we note that

$$i\omega\mu\mathbf{H} = \nabla \times \mathbf{E}.$$

Therefore

$$i\omega\mu H_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \qquad i\omega\mu H_y = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \qquad i\omega\mu H_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}.$$

From (8.1) and the expressions between (7.9) and (7.10),

$$\left.\begin{aligned} H_x &= \tfrac{1}{3}\pi k^3\sqrt{\epsilon/\mu}\,w(\xi,\eta)\sin 2\varphi,\\ H_y &= -\tfrac{1}{3}\pi k^3\sqrt{\epsilon/\mu}\,[u(\xi,\eta) + w(\xi,\eta)\cos 2\varphi],\\ H_z &= -\tfrac{1}{3}\pi k^3\sqrt{\epsilon/\mu}\,v(\xi,\eta)\sin\varphi, \end{aligned}\right\} \tag{8.13}$$

where

$$\left.\begin{aligned} iku(\xi,\eta) &= \frac{1}{\xi+\eta}(\xi F_\xi - \eta F_\eta) + \frac{\frac{1}{2}\sqrt{\xi\eta}}{\xi+\eta}(G_\xi + G_\eta) + \frac{1}{4\sqrt{\xi\eta}}G,\\ ikv(\xi,\eta) &= \frac{\sqrt{\xi\eta}}{\xi+\eta}(F_\xi + F_\eta - H_\xi - H_\eta) - \frac{1}{\sqrt{\xi\eta}}H,\\ ikw(\xi,\eta) &= \frac{1}{\xi+\eta}(\xi H_\xi - \eta H_\eta) + \frac{\frac{1}{2}\sqrt{\xi\eta}}{\xi+\eta}(G_\xi + G_\eta) - \frac{1}{4\sqrt{\xi\eta}}G, \end{aligned}\right\} \tag{8.14}$$

F, G, H being given in (8.2).

From (8.2) and (5.27)–(5.31),

$$\frac{\xi F_\xi - \eta F_\eta}{ik(\xi+\eta)} = \sum_0^\infty{}_n \left\{ \tfrac{1}{2}(n+1)^2(n^2+2n+6)V_n(2ik\xi) + \left[A_n + 2\sum_{n+1}^\infty{}_p A_p \right] S_n(2ik\xi) \right\} S_n(-2ik\eta),$$

$$\frac{\sqrt{\xi\eta}}{ik(\xi+\eta)}(F_\xi + F_\eta) = i\sum_0^\infty{}_n \left\{ \tfrac{1}{2}(n+2)(n^2+3n+6)V_n^1(2ik\xi) + (n+1)^{-1}\left[2\sum_{n+1}^\infty{}_p A_p \right] S_n^1(2ik\xi) \right\} S_n^1(-2ik\eta),$$

$$\frac{\sqrt{\xi\eta}}{2ik(\xi+\eta)}(G_\xi + G_\eta) + \frac{1}{4ik\sqrt{\xi\eta}}G = -\sum_0^\infty{}_n \left\{ \tfrac{1}{2}n(n+1)^2(n+2)V_n(2ik\xi) + \left[2\sum_n^\infty{}_p B_p \right] S_n(2ik\xi) \right\} S_n(-2ik\eta),$$

$$\frac{\sqrt{\xi\eta}}{2ik(\xi+\eta)}(G_\xi + G_\eta) - \frac{1}{4ik\sqrt{\xi\eta}}G = -\sum_{n+1}^\infty{}_n \left\{ \tfrac{1}{2}(n+2)(n+3)V_n^2(2ik\xi) + (n+1)^{-1}(n+2)^{-2}\left[2\sum_{n+1}^\infty{}_p B_p \right] S_n^2(2ik\xi) \right\} S_n^2(-2ik\eta),$$

$$\frac{\sqrt{\xi\eta}}{ik(\xi+\eta)}(H_\xi + H_\eta) + \frac{1}{ik\sqrt{\xi\eta}}H = i\sum_0^\infty{}_n \left\{ \tfrac{1}{2}n(n+2)(n+3)V_n^1(2ik\xi) + (n+1)^{-1}\left[2\sum_0^\infty{}_p C_p \right] S_n^1(2ik\xi) \right\} S_n^1(-2ik\eta),$$

$$\frac{\xi H_\xi - \eta H_\eta}{ik(\xi+\eta)} = \sum_0^\infty{}_n \left\{ \tfrac{1}{2}(n+2)(n+3)V_n^2(2ik\xi) + (n+1)^{-1}(n+2)^{-1}\left[C_n + 2\sum_{n+1}^\infty{}_p C_p \right] S_n^2(2ik\xi) \right\} S_n^2(-2ik\eta).$$

From these

$$\left.\begin{aligned} u(\xi,\eta) &= \sum_0^\infty{}_n \{3(n+1)^2 V_n(2ik\xi) + D_n S_n(2ik\xi)\} S_n(-2ik\eta), \\ v(\xi,\eta) &= i\sum_0^\infty{}_n \{3(n+2)V_n^1(2ik\xi) + (n+1)^{-1}2E_n S_n^1(2ik\xi)\} \cdot S_n^1(-2ik\eta), \\ w(\xi,\eta) &= \sum_0^\infty{}_n (n+1)^{-1}(n+2)^{-1}F_n S_n^2(2ik\xi)S_n^2(-2ik\eta), \end{aligned}\right\} \quad (8.15)$$

where

$$\left.\begin{aligned} D_n &= A_n + 2\sum_{n+1}^{\infty}{}_p A_p - 2\sum_{n}^{\infty}{}_p B_p, \\ E_n &= \sum_{n+1}^{\infty}{}_p A_p - \sum_{n}^{\infty}{}_p C_p, \\ F_n &= C_n + 2\sum_{n+1}^{\infty}{}_p C_p - 2\sum_{n+1}^{\infty}{}_p B_p. \end{aligned}\right\} \tag{8.16}$$

From (8.10)–(8.12),

$$D_n = E_n - \tfrac{3}{2}n(n+1)\frac{N_n(2ik\xi_0)}{M_n(2ik\xi_0)} + \frac{3i}{2\pi}\frac{n^2(n+1)(n+2)}{S_n(2ik\xi_0)M_n(2ik\xi_0)}, \tag{8.17}$$

$$E_n = 3\sum_{n+1}^{\infty}{}_p (p+1)\frac{N_p(2ik\xi_0)}{M_p(2ik\xi_0)} + \frac{3i}{2\pi}\sum_{n+1}^{\infty}{}_p \frac{p^2(p+1)}{S_p(2ik\xi_0)M_p(2ik\xi_0)}, \tag{8.18}$$

$$F_{n-1} = -E_{n-1} - \tfrac{3}{2}n(n+1)\frac{N_n(2ik\xi_0)}{M_n(2ik\xi_0)} + \frac{3i}{2\pi}\frac{n^2(n+1)(n+2)}{S_n(2ik\xi_0)M_n(2ik\xi_0)}. \tag{8.19}$$

To normalize these results it is necessary to compute the power radiated by our primary radiator. If this power is denoted by P_0, the fields due to a primary radiator of power P are obtained by multiplying the above fields by P/P_0.

To compute P_0, we will compute the power transmitted through a small sphere centered at the origin. From [2; §2.20].

$$P_0 = \int_0^{\pi}\int_0^{2\pi} S_r r^2 \sin\theta d\theta\, d\varphi = \tfrac{1}{2}Re\int_0^{\pi}\int_0^{2\pi} (E_\theta \tilde{H}_\varphi - E_\varphi \tilde{H}_\theta) r^2 \sin\theta d\theta\, d\varphi.$$

Since r may become as small as we please, we will be interestd only in terms of the order r^{-2} in $E_\theta\tilde{H}_\varphi - E_\varphi\tilde{H}_\theta$. Lower powers of r will cancel, and higher powers will go to zero.

To carry through the computation we will set up a spherical coordinate system with axis coincident with that of the dipole. The Hertz vector is then given by

$$\Pi = \mathbf{i}_r \psi\cos\theta - \mathbf{i}_\theta \psi \sin\theta,$$

where

$$\psi = [r^{-1} - r^{-1}(1 + ik^{-1}r^{-1})\sin\theta\sin\varphi]e^{ikr}$$

Now

$$\mathbf{E} = \nabla(\nabla\cdot\Pi) + k^2\Pi, \qquad \mathbf{H} = -i\omega\epsilon\nabla\times\Pi.$$

$$\phi = \nabla\cdot\Pi = \psi_r\cos\theta - r^{-1}\psi_\theta\sin\theta - r^{-1}\psi\cos\theta.$$

$$E_\theta = r^{-1}\phi_\theta - k^2\psi\sin\theta, \qquad E_\varphi = \phi_\varphi/(r\sin\theta),$$

$$H_\varphi = i\omega\epsilon[\psi_r\sin\theta + r^{-1}\psi_\theta\cos\theta], \qquad H_\theta = -i\omega\epsilon r^{-1}\cot\theta\psi_\varphi.$$

Dropping those terms which contribute nothing,

$$P_0 = -\tfrac{1}{2}\omega\epsilon k^2 Im \int_0^{\pi}\int_0^{2\pi} r^2\psi\bar{\psi}_r \sin^3\theta d\theta\, d\varphi$$

$$= \tfrac{1}{2}\omega\epsilon k^3 \int_0^{\pi}\int_0^{2\pi} (1 - \sin\theta\sin\varphi)^2 \sin^3\theta d\theta\, d\varphi.$$

Integrating this,

$$P_0 = \tfrac{28}{15}\pi\omega\epsilon k^3 = \tfrac{28}{15}\pi k\sqrt{\epsilon/\mu} \qquad (8.20)$$

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ALTERNATIVE FORMULAS FOR DIRECT INTERPOLATION OF A COMPLEX FUNCTION TABULATED ALONG EQUIDISTANT CIRCULAR ARCS

By Herbert E. Salzer

In a previous paper,[1] the author gave formulas for complex Lagrangian interpolation when one has an analytic function tabulated along equidistant circular arcs, for the 3-, 4-, and 5-point cases. In the 5-point case, the expressions were rather lengthy functions, involving the angular interval θ as a parameter, and their further extension to more than 5 points seemed to be too cumbersome to be practical, even with the understanding that for extensive use for some fixed θ those expressions were to be obtained beforehand in terms of that numerical value.

The present article furnishes a much more expedient method for direct interpolation and formulas are given here as far as the 9-point case.[2] This alternative method for direct interpolation suggested itself as a natural generalization of a scheme described in a recent paper by W. J. Taylor.[3] His method for n points $z_0, z_1, \cdots, z_{n-1}$ at which the functional values are $f_0, f_1, \cdots, f_{n-1}$ is as follows:

$$(1)\quad f \sim \left(\sum_{k=0}^{n-1} a_k f_k\right) \Big/ \sum_{k=0}^{n-1} a_k, \text{ where } a_k \equiv (-1)^{n-k-1}\binom{n-1}{k} \Big/ (p-k).$$

(Here $p = (z - z_0)/h$, where h is the tabular interval.) In his paper, (1) is applied only to real functions, but before proceeding to the main point of this article, it is interesting and useful to note that (1) holds intact for complex interpolation of an analytic function when the points z_k lie at uniform intervals along any straight line in the complex plane, where h now denotes that uniform complex interval, so that p is in general complex, since z will in general be off the line containing the points z_k. Of course, to employ (1) in this case, just as it stands, the notation $z_0, z_1, \cdots, z_{n-1}$ must remain unchanged.

Now Taylor's method can be generalized to facilitate interpolation, *real* or *complex*, provided that the reference points z_k are the same. Equation (8) in his paper, without the $n!$ which is used to obtain binomial coefficients and without the h, would express the Lagrangian approximation polynomial in the form

$$(z - z_0)(z - z_1) \cdots (z - z_{n-1}) \sum_{k=0}^{n-1} \{1/(z_k - z_0) \cdots (z_k - z_{k-1})(z_k - z_{k+1}) \cdots (z_k - z_{n-1})\} f_k/(z - z_k).$$

[1] "Formulas for Direct and Inverse Interpolation of a Complex Function Tabulated Along Equidistant Circular Arcs," Jour. of Math. and Phys., *24*, 1945, pp. 141–143.

[2] Formulas for the 10 and 11-point cases which are in the author's possession are not included in this article.

[3] "Method of Lagrangian Curvilinear Interpolation", Jour. Res. Nat. Bur. Stand., vol. 35, Aug. 1945, RP 1667 pp. 151–155. Full credit for suggesting the improvement over my previous method is due to Taylor's paper wherein is developed a simple but ingenious method of interpolating for a real function tabulated at equidistant intervals.

Then, just as in (1), one has

$$(2)\quad f \sim \left(\sum_{k=0}^{n-1} a_k f_k\right) \Big/ \sum_{k=0}^{n-1} a_k, \text{ but here}$$

$$a_k = 1/(z_k - z_0) \cdots (z_k - z_{k-1})(z_k - z_{k+1}) \cdots (z_k - z_{n-1})(z - z_k).$$

Writing $a_k = A_k^{(n)}/(z - z_k)$, for the same set of points z_k the complex quantities $A_k^{(n)} \equiv 1/(z_k - z_0) \cdots (z_k - z_{k-1})(z_k - z_{k+1}) \cdots (z_k - z_{n-1})$ can be found once and for all, and these would correspond to $(-1)^{n-k-1}\binom{n-1}{k}$ in Taylor's paper. Given the numbers $A_k^{(n)}$, the user need only divide by $z - z_k$ (just as in Taylor's method, except that now all divisions and subsequent operations involve complex quantities instead of real numbers) to obtain a_k for (2).

In the present paper, the above principle is applied to direct interpolation of an analytic function of z which can be approximated by a polynomial of degree $n - 1$ and which is tabulated at equally spaced intervals along the arc of a circle in the z-plane. Coefficients $A_k^{(n)}$ are given below as functions of θ only. As in the previous article,[1] for extensive use for some fixed θ, the quantities $A_k^{(n)}$ should be obtained as fixed complex numbers. Thus, the "length" of the expressions $A_k^{(n)}$ as a function of θ, in say the 9-point case, is really no deterrent to their practicality. When the number of points increases, the efficiency of this method becomes enormously greater than that in the author's original plan which employed the explicit Lagrangian coefficients. For, in this method the number of operations increases linearly with n, whereas in the other plan the number increases quadratically. In this method, after the quantities $A_k^{(n)}$ are found as numbers, 9-point interpolation is not much more laborious than 5-point.

This improvement applies only to *direct* interpolation. Of course, inverse interpolation by Lagrange's formula can be facilitated by Taylor's method if one replaces in (2) the f and f_k by z and z_k respectively and replaces z and every z_j in $a_k \equiv A_k^{(n)}/(z - z_k)$ by f and f_j respectively. In some particular cases where there are many inverse interpolations involving the same set of quantities f_k, it might be more advantageous to calculate the quantities $A_k^{(n)} \equiv 1/(f_k - f_0) \cdots (f_k - f_{k-1})(f_k - f_{k+1}) \cdots (f_k - f_{n-1})$ to be used repeatedly, than to employ the method suggested at the end of the previous paper.[1]

It is convenient to change the subscript notation to that usually employed in Lagrangian interpolation, so that in the n-point formula the index of summation, instead of going from 0 to $n - 1$, goes from $-[(n - 1)/2]$ to $[n/2]$, where [ ] is the "nearest integer" notation. Then by an obvious preliminary change of variable, if the points along the circle are $\cdots z_{-2}, z_{-1}, z_0, z_1, z_2, \cdots$, after the change of variable, keeping the notation unchanged, $z_1 - z_0 = 1$ and $z_0 = 0$. (See the second paragraph, p. 141 of the previous article,[1] only with the correction in the angle of rotation which is erroneously given as $\pi - \alpha$ instead of $\pi - \alpha - \frac{1}{2}\theta$. This preliminary change is equivalent to first performing the rotation and then taking the P in that paper as the new variable z.)

From (2) it is obvious that, for each n, one may multiply all the quantities $A_k^{(n)}$ by any non-zero factor. This will be done to obtain expressions for $A_k^{(n)}$ which are as simple as possible. To avoid too much notation, after each such multiplication, the new coefficients will still be denoted by $A_k^{(n)}$.

The diagram in the previous paper[1] is simplified to appear thus:

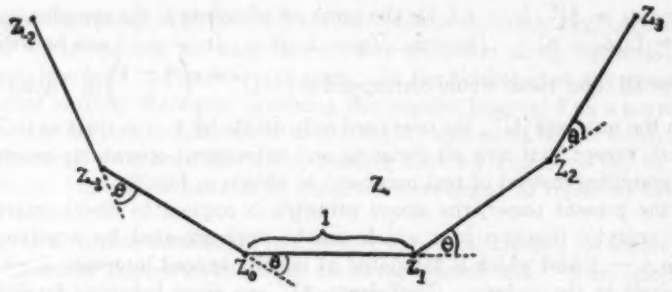

FIG. 1.

Evidently, $z_0 = 0$, $z_1 = 1$, $z_2 = 1 + e^{i\theta}$, $z_3 = 1 + e^{i\theta} + e^{2i\theta}$, $\cdots$, $z_m = 1 + e^{i\theta} + \cdots + e^{i(m-1)\theta}$, while $z_{-1} = -e^{-i\theta}$, $z_{-2} = -e^{-i\theta} - e^{-2i\theta}$, $\cdots$, $z_{-m} = -e^{-i\theta} - \cdots - e^{-im\theta}$. From the formula for summing a geometric series, it turns out that $z_k = (e^{ik\theta} - 1)/(e^{i\theta} - 1)$ for any k, positive, negative, or zero. Since $A_k^{(n)} = 1/(z_k - z_{-[(n-1)/2]}) \cdots (z_k - z_{k-1})(z_k - z_{k+1}) \cdots (z_k - z_{[n/2]})$, cancelling a common factor $(e^{i\theta} - 1)^{n-1}$ and using Π' to denote the product from $j = -[(n-1)/2]$ to $j = [n/2]$, omitting $j = k$, one obtains for $A_k^{(n)}$ the expression $1/\Pi'(e^{ik\theta} - e^{ij\theta})$, or $\Pi'(e^{-ik\theta} - e^{-ij\theta})/\Pi'(2 - 2\cos(k-j)\theta)$, from which the factor 2^{n-1} in the denominator can be removed.[4]

For n odd, it is readily shown that for any k, including 0,

$$A_{-k}^{(n)} = \overline{A_k^{(n)}}, \tag{3}$$

where the bar denotes the complex conjugate. To prove (3) it suffices to consider $A_{-k}^{(n)}$ in its next to the original form, i.e. $1/\Pi'(e^{-ik\theta} - e^{ij\theta})$, after the $(e^{i\theta} - 1)^{n-1}$ has been cancelled, because $\overline{AB} = A\overline{B}$ when A is real. Since $\overline{(1/A)} = 1/\overline{A}$, instead of proving (3) for $A_{-k}^{(n)}$ and $A_k^{(n)}$, it suffices to prove (3) for $1/A_{-k}^{(n)}$ and $1/A_k^{(n)}$ where $1/A_{-k}^{(n)} \equiv \Pi'(e^{-ik\theta} - e^{ij\theta})$. But it is apparent that in this last product, for every factor with a positive (negative) j there is a corresponding conjugate factor in $1/A_k^{(n)}$ with a negative (positive) j. Hence (3) is established.

For n even, the following useful relation exists:

$$A_{k+1}^{(n)} = -\exp[i\theta\{k(2-n) - \tfrac{1}{2}n + 1\}]A_{-k}^{(n)}. \tag{4}$$

To prove (4), it suffices to establish it for $P_{-k}^{(n)} \equiv \Pi'(e^{ik\theta} - e^{-ij\theta})$ and $P_{k+1}^{(n)}$, because the denominators of $A_{-k}^{(n)}$ and $A_{k+1}^{(n)}$ are the same. Each factor of $P_{-k}^{(n)}$,

[4] In order that $(1 - \cos(k-j)\theta) \neq 0$, $\theta \neq 2m\pi/(k-j)$. It certainly suffices, in the n point case, if $\theta < 2\pi/(n-1)$.

namely $(e^{j\theta} - e^{-j\theta})$, when multiplied by $-e^{i(-k+j-1)\theta}$, becomes $(e^{-i(k+1)\theta} - e^{-i(1-j)\theta})$. When j runs from $-[(n-1)/2]$ to $\frac{1}{2}n$, leaving out $-k$, it is also true that $1-j$ runs from $\frac{1}{2}n$ to $-[(n-1)/2]$, leaving out $k+1$. Thus the corresponding factors of $P_{k+1}^{(n)}$ are obtained. Hence $P_{k+1}^{(n)} = (-1)^{n-1}\exp\{i\theta\sum'(-k+j-1)\}P_{-k}^{(n)}$, where $\sum'$ is for the above described range of j. Since $\sum'(-k-1) = (n-1)(-k-1)$ and $\sum' j = \frac{1}{2}n + k$, $P_{k+1}^{(n)} = -e^{i\theta[\frac{1}{2}(3-n)-\frac{1}{2}n+1]}P_{-k}^{(n)}$, which proves (4).

As a further simplification, for every n, the quantities $A_k^{(n)}$ are multiplied through by the least common multiple of the denominators, namely, $(1-\cos\theta)^2 \cdots (1-\cos[(n-1)/2]\theta)^2(1-\cos[(n+1)/2]\theta) \cdots (1-\cos(n-1)\theta)$. Finally, for n odd, the factor $1-\cos\frac{1}{2}(n-1)\theta$ is cancelled out, and for n even, by writing $A_0^{(n)}$ in a certain way, the factor $1-\cos(\frac{1}{2}n-1)\theta$ is cancelled out. These last simplifications obviously do not affect the relations (3) and (4), which continue to hold for the coefficients $A_k^{(n)}$ in this new form. (An additional simplification is the omission of the factor $2\sin\theta$ from the 3-point coefficients and the factor 2 from the 4-point coefficients.)

TABLE OF COEFFICIENTS

3-point

$$A_{-1}^{(3)} = -\sin\theta + i(\cos\theta - 1), \quad A_0^{(3)} = 2\sin\theta, \quad A_1^{(3)} = \overline{A_{-1}^{(3)}}.$$

4-point

$$A_{-1}^{(4)} = i(\sin 3\theta - \sin 2\theta - \sin\theta), \quad A_0^{(4)} = (1-\cos 3\theta)[(-\cos 2\theta + 1) + i\sin 2\theta],$$
$$A_1^{(4)} = (-\cos\theta + i\sin\theta)A_0^{(4)}, \quad A_2^{(4)} = (-\cos 3\theta + i\sin 3\theta)A_{-1}^{(4)}.$$

5-point

$$A_{-2}^{(5)} = (1-\cos\theta)[(\cos 8\theta - \cos 7\theta - \cos 6\theta + 2\cos 3\theta + \cos 2\theta - \cos\theta - 1) + i(\sin 8\theta - \sin 7\theta - \sin 6\theta + 2\sin 3\theta - \sin 2\theta + \sin\theta)],$$
$$A_{-1}^{(5)} = (1-\cos 4\theta)[(-\cos 5\theta + 2\cos 4\theta - 2\cos 2\theta + \cos\theta) + i(-\sin 5\theta + 2\sin 4\theta - 3\sin\theta)],$$
$$A_0^{(5)} = 4(1-\cos 4\theta)(1-\cos 3\theta)(1-\cos\theta), \quad A_1^{(5)} = \overline{A_{-1}^{(5)}}, \quad A_2^{(5)} = \overline{A_{-2}^{(5)}}.$$

6-point

$$A_{-2}^{(6)} = (1-\cos\theta)[(\cos 10\theta - \cos 9\theta - \cos 8\theta + 2\cos 4\theta + 2\cos 3\theta - \cos 2\theta - \cos\theta - 1) + i(\sin 10\theta - \sin 9\theta - \sin 8\theta + 2\sin 5\theta - \sin 2\theta - \sin\theta)],$$
$$A_{-1}^{(6)} = (1-\cos 5\theta)[(-\cos 6\theta + 3\cos 5\theta - 2\cos 4\theta - \cos 3\theta + \cos 2\theta - 2\cos\theta + 2) + i(-\sin 6\theta + \sin 5\theta + 2\sin 4\theta - \sin 3\theta - \sin 2\theta - 2\sin\theta)],$$
$$A_0^{(6)} = 4(1-\cos 5\theta)(1-\cos 4\theta)(1-\cos\theta)[(-\cos 3\theta + 1) + i\sin 3\theta)],$$
$$A_1^{(6)} = (-\cos 2\theta + i\sin 2\theta)A_0^{(6)}, \quad A_2^{(6)} = (-\cos 6\theta + i\sin 6\theta)A_{-1}^{(6)},$$
$$A_3^{(6)} = (-\cos 10\theta + i\sin 10\theta)A_{-2}^{(6)}.$$

7-point

$$A_{-3}^{(7)} = (1-\cos 2\theta)(1-\cos\theta)[(\cos 18\theta - \cos 17\theta - \cos 16\theta + \cos 13\theta + 2\cos 11\theta - \cos 9\theta - \cos 8\theta - \cos 7\theta - \cos 6\theta + 2\cos 4\theta + \cos 3\theta - \cos\theta) + i(\sin 18\theta - \sin 17\theta - \sin 16\theta + \sin 13\theta + 2\sin 11\theta - \sin 9\theta - \sin 8\theta - \sin 7\theta - \sin 6\theta + 2\sin 4\theta - \sin 3\theta + 2\sin 2\theta + \sin\theta)],$$
$$A_{-2}^{(7)} = (1-\cos 6\theta)(1-\cos 2\theta)[(-\cos 13\theta + 2\cos 12\theta - \cos 10\theta -$$

$$\cos 8\theta + 2\cos 5\theta - \cos 3\theta + \cos 2\theta - 1) + i(-\sin 13\theta + 2\sin 12\theta - \sin 10\theta - \sin 8\theta + 2\sin 5\theta + \sin 3\theta - 3\sin 2\theta)],$$

$$A_{-1}^{(7)} = (1 - \cos 6\theta)(1 - \cos 5\theta)[(\cos 9\theta - 2\cos 8\theta - \cos 7\theta + 3\cos 6\theta + 2\cos 4\theta - 4\cos 3\theta - 3\cos 2\theta + 4\cos\theta) + i(\sin 9\theta - 2\sin 8\theta - \sin 7\theta + 3\sin 6\theta - \sin 2\theta - 2\sin\theta)],$$

$$A_0^{(7)} = 8(1 - \cos 6\theta)(1 - \cos 5\theta)(1 - \cos 4\theta)(1 - \cos 2\theta)(1 - \cos\theta),$$

$$A_1^{(7)} = \overline{A_{-1}^{(7)}},\ A_2^{(7)} = \overline{A_{-2}^{(7)}},\ A_3^{(7)} = \overline{A_{-3}^{(7)}}.$$

8-point

$$A_{-3}^{(8)} = (1 - \cos 2\theta)(1 - \cos\theta)[(\cos 21\theta - \cos 20\theta - \cos 19\theta + \cos 16\theta + \cos 14\theta + \cos 13\theta - \cos 11\theta - \cos 10\theta - 2\cos 9\theta - \cos 7\theta + \cos 6\theta + 3\cos 5\theta + \cos 4\theta + \cos 3\theta - \cos 2\theta - \cos\theta - 1) + i(\sin 21\theta - \sin 20\theta - \sin 19\theta + \sin 16\theta + \sin 14\theta + \sin 13\theta - \sin 11\theta - \sin 10\theta - 2\sin 9\theta + \sin 7\theta - \sin 6\theta + \sin 5\theta + \sin 4\theta + \sin 3\theta + \sin 2\theta - \sin\theta)],$$

$$A_{-2}^{(8)} = (1 - \cos 7\theta)(1 - \cos 2\theta)[(-\cos 15\theta + 2\cos 14\theta - \cos 12\theta - \cos 10\theta + \cos 9\theta - 2\cos 8\theta + 3\cos 7\theta - \cos 6\theta + \cos 4\theta - 3\cos\theta + 2) + i(-\sin 15\theta + 2\sin 14\theta - \sin 12\theta - \sin 10\theta + \sin 9\theta - 2\sin 8\theta + \sin 7\theta + 3\sin 6\theta - \sin 4\theta - 2\sin 2\theta - \sin\theta)],$$

$$A_{-1}^{(8)} = (1 - \cos 7\theta)(1 - \cos 6\theta)[(\cos 10\theta - 2\cos 9\theta - 2\cos 8\theta + 5\cos 7\theta + \cos 6\theta - 3\cos 5\theta - \cos 3\theta - 2\cos 2\theta + \cos\theta + 2) + i(\sin 10\theta - 2\sin 9\theta + \sin 7\theta - \sin 6\theta + 3\sin 5\theta - \sin 3\theta - 2\sin 2\theta - \sin\theta)],$$

$$A_0^{(8)} = 8(1 - \cos 7\theta)(1 - \cos 6\theta)(1 - \cos 5\theta)(1 - \cos 2\theta)(1 - \cos\theta)\cdot[(-\cos 4\theta + 1) + i\sin 4\theta)],$$

$$A_1^{(8)} = (-\cos 3\theta + i\sin 3\theta)A_0^{(8)},\ A_2^{(8)} = (-\cos 9\theta + i\sin 9\theta)A_{-1}^{(8)},$$

$$A_3^{(8)} = (-\cos 15\theta + i\sin 15\theta)A_{-2}^{(8)},\ A_4^{(8)} = (-\cos 21\theta + i\sin 21\theta)A_{-3}^{(8)}.$$

9-point

$$A_{-4}^{(9)} = (1 - \cos 3\theta)(1 - \cos 2\theta)(1 - \cos\theta)[(\cos 32\theta - \cos 31\theta - \cos 30\theta + \cos 27\theta + \cos 25\theta + \cos 23\theta - \cos 21\theta - 2\cos 20\theta - \cos 19\theta - \cos 17\theta + \cos 16\theta + \cos 15\theta + 2\cos 14\theta + \cos 13\theta + \cos 12\theta - \cos 11\theta - \cos 9\theta - 2\cos 8\theta - \cos 7\theta + \cos 5\theta + \cos 4\theta - \cos 2\theta + \cos\theta) + i(\sin 32\theta - \sin 31\theta - \sin 30\theta + \sin 27\theta + \sin 25\theta + \sin 23\theta - \sin 21\theta - 2\sin 20\theta - \sin 19\theta - \sin 17\theta + \sin 16\theta + \sin 15\theta + 2\sin 14\theta + \sin 13\theta + \sin 12\theta - \sin 11\theta - \sin 9\theta - 2\sin 8\theta - \sin 7\theta + \sin 5\theta - \sin 4\theta + 2\sin 3\theta + \sin 2\theta + \sin\theta)],$$

$$A_{-3}^{(9)} = (1 - \cos 8\theta)(1 - \cos 3\theta)(1 - \cos 2\theta)[(-\cos 25\theta + 2\cos 24\theta - \cos 22\theta - \cos 20\theta + \cos 19\theta - \cos 18\theta + \cos 16\theta + \cos 15\theta + \cos 13\theta - 2\cos 12\theta - 2\cos 9\theta + \cos 8\theta + \cos 6\theta + \cos 5\theta - \cos 4\theta + \cos 3\theta + \cos 2\theta - 2\cos\theta) + i(-\sin 25\theta + 2\sin 24\theta - \sin 22\theta - \sin 20\theta + \sin 19\theta - \sin 18\theta + \sin 16\theta + \sin 15\theta + \sin 13\theta - 2\sin 12\theta - 2\sin 9\theta + \sin 8\theta + \sin 6\theta + \sin 5\theta + \sin 4\theta - 3\sin 3\theta + \sin 2\theta)],$$

$$A_{-2}^{(9)} = (1 - \cos 8\theta)(1 - \cos 7\theta)(1 - \cos 3\theta)[(\cos 19\theta - 2\cos 18\theta - \cos 17\theta + 3\cos 16\theta - \cos 13\theta + \cos 12\theta - \cos 11\theta - 3\cos 10\theta + 2\cos 9\theta + \cos 8\theta + \cos 6\theta + 3\cos 5\theta - 5\cos 4\theta - 2\cos 3\theta + 4\cos 2\theta - \cos\theta) + i(\sin 19\theta - 2\sin 18\theta - \sin 17\theta + 3\sin 16\theta - \sin 13\theta + \sin 12\theta -$$

$$\sin 11\theta - 3\sin 10\theta + 2\sin 9\theta + \sin 8\theta + \sin 6\theta + \sin 5\theta - \sin 4\theta - 2\sin 2\theta - \sin\theta)],$$

$$A_{-1}^{(9)} = (1-\cos 8\theta)(1-\cos 7\theta)(1-\cos 6\theta)[(-\cos 14\theta + 2\cos 13\theta + \cos 12\theta - 2\cos 11\theta - 2\cos 10\theta - \cos 9\theta + 3\cos 8\theta + 4\cos 6\theta + \cos 5\theta - 5\cos 4\theta - 5\cos 3\theta - \cos 2\theta + 5\cos\theta + 1) + i(-\sin 14\theta + 2\sin 13\theta + \sin 12\theta - 2\sin 11\theta - 2\sin 10\theta - \sin 9\theta + 3\sin 8\theta + 2\sin 7\theta - \sin 5\theta - \sin 4\theta - \sin 3\theta + \sin 2\theta - \sin\theta)],$$

$$A_0^{(9)} = 16(1-\cos 8\theta)(1-\cos 7\theta)(1-\cos 6\theta)(1-\cos 5\theta)(1-\cos 3\theta)(1-\cos 2\theta)(1-\cos\theta),$$

$$A_1^{(9)} = \overline{A_{-1}^{(9)}}, A_2^{(9)} = \overline{A_{-2}^{(9)}}, A_3^{(9)} = \overline{A_{-3}^{(9)}}, A_4^{(9)} = \overline{A_{-4}^{(9)}}.$$

MATHEMATICAL TABLES PROJECT
NATIONAL BUREAU OF STANDARDS

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ON CURVED SHOCK WAVES

By T. Y. Thomas

1. Introduction. We deal with the stationary flow of an ideal gas with viscosity and thermal conductivity neglected. If a shock wave arises, e.g. as the result of the presence of an obstacle in the field of flow, it will be assumed that this divides the flow under consideration into a region 1 and a region 2. We suppose region 1 to be traversed by the gas before contact with the shock surface (shock line in the case of plane flow) and region 2 to be that into which the gas enters after passing through the shock surface. It is assumed that this surface is given by a continuous and differentiable function of the coordinates and that it has a continuous and differentiable unit normal ν which we suppose directed from region 1 into region 2. Furthermore we assume the required differentiability conditions to be satisfied by the velocity components u_α, the density ρ and pressure p in each region 1 and 2 so that it is possible to express the directional derivatives of these quantities along either side of the shock surface by means of the usual formulas involving partial differentiation.

Denote by $u_{1\alpha}$, ρ_1 and p_1 the velocity components, density and pressure on the side of the wave bordering region 1 and correspondingly by $u_{2\alpha}$, ρ_2 and p_2 the values of these quantities on the side of the wave bordering the region 2. Put $u_{1n} = u_{1\alpha}\nu^\alpha$ and $u_{2n} = u_{2\alpha}\nu^\alpha$ so that u_{1n} and u_{2n} are the normal components of velocity on sides 1 and 2 of the wave surface. Then the shock conditions for the stationary wave under consideration can be written in the form

$$[u_\alpha] = \frac{2(\rho_1 u_{1n}^2 - \gamma p_1)\nu_\alpha}{-(\gamma+1)\rho_1 u_{1n}}, \tag{1}$$

$$[p] = \frac{2(\rho_1 u_{1n}^2 - \gamma p_1)}{\gamma+1}, \tag{2}$$

$$[\rho] = \frac{2\rho_1(\rho_1 u_{1n}^2 - \gamma p_1)}{2\gamma p_1 + (\gamma-1)\rho_1 u_{1n}^2}. \tag{3}$$

Here the bracket [ ] denotes the difference of the values on the two sides of the shock surface of the quantity enclosed, e.g. $[u_\alpha] = u_{2\alpha} - u_{1\alpha}$, and γ stands for the ratio of the two specific heats c_p and c_v, i.e. $\gamma = c_p/c_v$. It is assumed that γ is constant throughout the flow.

We need also to indicate the differential relations governing the flow.[1] Referred to rectangular coordinates these are

$$p_{,\alpha} + \rho u_\sigma u_{\alpha,\sigma} = 0, \tag{4}$$

[1] It is to be understood here, and in the following, that an index which occurs twice in a term is to be summed over the admissable values of the index. Since there is no distinction between covariant and contravariant indices within a rectangular system we may write any index as a superscript or subscript without modification of the value of the term in which the index appears.

$$\rho_{,\alpha}u_\alpha + \rho u_{\sigma,\sigma} = 0, \tag{5}$$

$$\rho u_{\alpha,\beta}u_\alpha u_\beta - \gamma p u_{\sigma,\sigma} = 0. \tag{6}$$

Equations (4) are the usual equations for the motion of a fluid without viscosity. (5) is the equation of continuity. Equation (6) is the hydrodynamical equivalent of the condition that the entropy be constant along a stream line.

The object of this paper is to obtain formulas for the determination of the derivatives $u_{\alpha,\beta}$, $\rho_{,\alpha}$ and $p_{,\alpha}$ behind the shock surface, i.e. on the side of the surface bordering the region 2, under the assumption that the flow before the surface, i.e. in region 1, is known. This problem presents no particular difficulty. Our essential contribution therefore consists in devising a technical procedure by which these derivatives are easily obtained and the formulas for them appear as rather compact expressions. In our calculations we have confined our attention to the case of plane or two dimensional flow. However the method can be applied to three dimensional flow as well as to the determination of higher derivatives of the velocity, density and pressure behind the shock surface.

2. **Differentiation of shock relations.** In the case of plane flow let λ be the unit tangent vector to the shock line and suppose λ so directed that the perpendicular vectors λ, ν form a right handed system. We then have the relations $\lambda^1 = \nu^2$, $\lambda^2 = -\nu^1$ or more briefly $\lambda_\alpha = e_{\alpha\beta}\nu^\beta$ where $e_{12} = -e_{21} = 1$ and $e_{11} = e_{22} = 0$.

Let s denote arc length along the shock line; we choose s to increase algebraically in the direction specified by the above vector λ. Between the directional derivatives of the vectors λ and ν we will then have the following relations $d\lambda^\alpha/ds = \kappa\nu^\alpha$, $d\nu^\alpha/ds = -\kappa\lambda^\alpha$ where k is the curvature of the shock line. For vectors λ, ν and arc length s as chosen the curvature κ will be non-negative.

Now differentiate the relations (1), (2) and (3) with respect to s. This gives relations along the shock line of the form

$$u_{\alpha,\beta}\lambda^\beta = u_{1\alpha,\beta}\lambda^\beta + A_\alpha = A^*_\alpha, \tag{7}$$

$$p_{,\beta}\lambda^\beta = p_{1,\beta}\lambda^\beta + B = B^*, \tag{8}$$

$$\rho_{,\beta}\lambda^\beta = \rho_{1,\beta}\lambda^\beta + C = C^*, \tag{9}$$

where for simplicity we have omitted the subscript 2 on the quantities appearing in the left members of these equations. The explicit formulas for the quantities A_α, B and C are readily obtained by differentiation of the right members of (1), (2) and (3) respectively; we see that these quantities are expressible in terms of the $u_{1\alpha}$, ρ_1 and p_1 and their partial derivatives along the shock line, the vectors λ and ν, and the curvature κ. For brevity in the following any quantity which can so be expressed will be said to have an *allowable determination*.

3. **Determination of the partial derivatives behind the shock line.** Consider the three equations (4), (5), (6) and also the three equations (7), (8) and (9).

From this set of six equations we can determine the partial derivatives of the velocity components u_α, the density ρ and the pressure p immediately behind the shock line. For this purpose we first eliminate the derivatives $p_{,\alpha}$ to obtain the following equations, which we represent as two sets of equations for convenience of reference, namely,

$$\rho_{,\beta}\lambda^\beta = C^*, \qquad \rho_{,\beta}u^\beta + \rho u_{\sigma,\sigma} = 0, \tag{10}$$

and

$$u_{\alpha,\beta}\lambda^\beta = A^*_\alpha, \qquad \rho u_{\alpha,\beta}\lambda^\alpha u^\beta = -B^*, \qquad \rho u_{\alpha,\beta}u^\alpha u^\beta - \gamma p u_{\sigma,\sigma} = 0. \tag{11}$$

In fact from (11) we can determine the $u_{\alpha,\beta}$ and then from (10) the $\rho_{,\beta}$ can be found; after this, equations (4) yield the values of the derivatives $p_{,\alpha}$.

To effect the above determinations let us put $\xi^\alpha_1 = \lambda^\alpha$, $\xi^\alpha_2 = u_\alpha$ or in matrix form

$$\| \xi^\alpha_i \| = \begin{Vmatrix} \xi^1_1 & \xi^2_1 \\ \xi^1_2 & \xi^2_2 \end{Vmatrix} = \begin{Vmatrix} \lambda^1 & \lambda^2 \\ u^1 & u^2 \end{Vmatrix}.$$

Hence determinant $| \xi^\alpha_i | = u_2\lambda^1 - u_1\lambda^2 = u_\alpha\nu^\alpha = u_n$ where u_n now denotes the normal component of velocity immediately behind the shock line. Also define quantities ζ^i_α as follows

$$\zeta^i_\alpha = \text{cofactor of } \xi^\alpha_i \text{ in } | \xi^\alpha_i | / | \xi^\alpha_i |,$$

so that

$$\| \zeta^i_\alpha \| = \begin{Vmatrix} \zeta^1_1 & \zeta^2_1 \\ \zeta^1_2 & \zeta^2_2 \end{Vmatrix} = \begin{Vmatrix} u^2/u_n & -\lambda^2/u_n \\ -u^1/u_n & \lambda^1/u_n \end{Vmatrix}.$$

The above matrices will enable us to see at a glance the values of the various components of ξ^α_i and ζ^i_α in the following discussion.

Now consider the expressions $u_{\alpha,\beta}\xi^\alpha_i\xi^\beta_j$. From (11) we have

$$\begin{cases} u_{\alpha,\beta}\xi^\alpha_1\xi^\beta_1 = u_{\alpha,\beta}\lambda^\alpha\lambda^\beta = A^*_\alpha\lambda^\alpha = A_{11}, & u_{\alpha,\beta}\xi^\alpha_1\xi^\beta_2 = u_{\alpha,\beta}\lambda^\alpha u^\beta = -B^*/\rho = A_{12}, \\ u_{\alpha,\beta}\xi^\alpha_2\xi^\beta_1 = u_{\alpha,\beta}u^\alpha\lambda^\beta = A^*_\alpha u^\alpha = A_{21}, & u_{\alpha,\beta}\xi^\alpha_2\xi^\beta_2 = u_{\alpha,\beta}u^\alpha u^\beta = (\gamma p/\rho)\, u_{\sigma,\sigma} = A_{22}, \end{cases} \tag{12}$$

where any quantity A_{ij} is defined by the quantity immediately preceding it in these relations. More briefly we can write

$$u_{\alpha,\beta}\xi^\alpha_i\xi^\beta_j = A_{ij}. \tag{13}$$

Then multiplying the members of (13) by $\xi^i_\sigma\zeta^j_\tau$ and summing on the repeated indices i and j we have

$$u_{\sigma,\tau} = A_{ij}\zeta^i_\sigma\zeta^j_\tau. \tag{14}$$

Now the first three relations (12) give an allowable determination of the quantities A_{11}, A_{12} and A_{21} (see end of §2). However the A_{22} is given in terms of the $u_{\sigma,\sigma}$ by the fourth equation (12); it remains therefore to effect an allow-

able determination of A_{22}. But this can readily be accomplished by contracting the indices σ and τ in (14). We thus find

$$u_{\sigma,\sigma} = A_{ij}\zeta^i_\sigma\zeta^j_\sigma = A_{11}\zeta^1_\sigma\zeta^1_\sigma + (A_{12} + A_{21})\zeta^1_\sigma\zeta^2_\sigma + A_{22}\zeta^2_\sigma\zeta^2_\sigma. \tag{15}$$

But,

$$\begin{cases} \zeta^1_\sigma\zeta^1_\sigma = \left(\dfrac{u^2}{u_n}\right)^2 + \left(\dfrac{u^1}{u_n}\right)^2 = \dfrac{u_\sigma u_\sigma}{u_n^2}, \\ \zeta^1_\sigma\zeta^2_\sigma = -\dfrac{u_2\lambda^2}{u_n^2} - \dfrac{u_1\lambda^1}{u_n^2} = -\dfrac{u_\alpha\lambda^\alpha}{u_n^2}, \\ \zeta^2_\sigma\zeta^2_\sigma = \left(\dfrac{\lambda^2}{u_n}\right)^2 + \left(\dfrac{\lambda^1}{u_n}\right)^2 = \dfrac{1}{u_n^2}. \end{cases}$$

Hence substituting these values in the expanded form of the right member of (15) we are led to the following allowable determination of A_{22}, namely

$$A_{22} = \frac{\gamma p}{\rho}\cdot\frac{(B^* - \rho A^*_\sigma u^\sigma)u_\alpha\lambda^\alpha + \rho u_\sigma u_\sigma A^*_\alpha\lambda^\alpha}{\rho u_n^2 - \gamma p}. \tag{16}$$

From the original definition of the A_{22} the factor after the "dot" in (16) is seen to give an allowable determination of the quantity $u_{\sigma,\sigma}$.

Now the set of two equations (10) can be written in the form $\rho_{,\beta}\xi^\beta_i = a_i$ where $a_1 = C^*$ and $a_2 = -\rho u_{\sigma,\sigma}$. Hence we have

$$\rho_{,\sigma} = a_i\zeta^i_\sigma \tag{17}$$

as the allowable determination of the derivatives $\rho_{,\sigma}$. Finally from (4) the derivatives $p_{,\alpha}$ behind the shock line are determined in an allowable manner by the equations

$$p_{,\alpha} = -\rho u_{\alpha,\beta}u^\beta = -\rho A_{ij}\zeta^i_\alpha\zeta^j_\beta\xi^\beta_2 = -\rho A_{i2}\zeta^i_\alpha. \tag{18}$$

4. **Calculation of the invariants a_i and A_{ij}.** Suppose that the flow in front of the shock line is *uniform*, e.g. the pressure, density and velocity vector are constant in the region 1. We now give explicit formulas for the invariants a_i and A_{ij} under this assumption.

Observe that

$$du_{1n}/ds = u_{1\alpha}dv^\alpha/ds = -u_{1\alpha}\lambda^\alpha\kappa = -u_t\kappa,$$

where u_t is the tangential component of velocity along the shock line. Also for uniform flow in the region 1 we have $A^*_\alpha = A_\alpha$, $B^* = B$ and $C^* = C$. Hence we readily find

$$A_\alpha = -\frac{Bv_\alpha}{\rho_1 u_{1n}} + \frac{[\rho]}{\rho}(u_{1n}\lambda_\alpha - u_t v_\alpha)\kappa, \tag{19}$$

$$B = -\frac{4\rho_1 u_{1n}u_t\kappa}{\gamma + 1}, \tag{20}$$

$$C = \frac{2u_t}{(\gamma+1)u_n}\{(\gamma-1)\rho - (\gamma+1)\rho_1\}\kappa. \tag{21}$$

Also

$$\rho A_\alpha \lambda^\alpha = u_{1n}[\rho]\kappa, \tag{22}$$

$$\rho A_\alpha u^\alpha = \left(\frac{[\rho]^2}{\rho_1\rho} + \frac{4}{\gamma+1}\right)\rho_1 u_{1n} u_t \kappa, \tag{23}$$

$$B - \rho A_\alpha u^\alpha = -\left(\frac{[\rho]^2}{\rho_1\rho} + \frac{8}{\gamma+1}\right)\rho_1 u_{1n} u_t \kappa. \tag{24}$$

Formulas (19), $\cdots$, (24) can be used in the calculation of the invariants a_i and A_{ij}. From the definition of the A_{11}, A_{12} and A_{21} contained in the first three equations (12) and from equation (16) for A_{22} we find

$$A_{11} = u_n[\rho]\kappa/\rho_1, \tag{25}$$

$$A_{12} = 4u_n u_t \kappa/(\gamma+1), \tag{26}$$

$$A_{21} = \left(\frac{[\rho]^2}{\rho_1\rho} + \frac{4}{\gamma+1}\right)u_n u_t \kappa, \tag{27}$$

$$A_{22} = \frac{\gamma p \mu_n \kappa}{\rho u_n^2 - \gamma p}\left\{\frac{v^2[\rho]}{\rho_1} - \left(\frac{[\rho]^2}{\rho_1\rho} + \frac{8}{\gamma+1}\right)u_t^2\right\}, \tag{28}$$

where v is the magnitude of the velocity behind the shock line. Correspondingly, using the relations $a_1 = C$ and $a_2 = -\rho u_{\sigma,\sigma}$, with $u_{\sigma,\sigma}$ given by the second factor in the right member of (16) as observed above, we have

$$a_1 = \frac{2u_t}{(\gamma+1)u_n}\{(\gamma-1)\rho - (\gamma+1)\rho_1\}\kappa, \tag{29}$$

$$a_2 = \frac{-\rho^2 u_n \kappa}{\rho u_n^2 - \gamma p}\left\{\frac{v^2[\rho]}{\rho_1} - \left(\frac{[\rho]^2}{\rho_1\rho} + \frac{8}{\gamma+1}\right)u_t^2\right\}. \tag{30}$$

5. **Variation of velocity, density and pressure along the shock line.** The rate of change with respect to the arc length s of the velocity components u_α, the density ρ and the pressure p along the shock line is given by the following formulas

$$du_\alpha/ds = u_{\alpha,\beta}\lambda^\beta = A_{ij}\zeta_\alpha^i \zeta_\beta^j \xi_1^\beta = A_{i1}\zeta_\alpha^i,$$

$$d\rho/ds = \rho_{,\alpha}\lambda^\alpha = a_i \zeta_\alpha^i \xi_1^\alpha = a_1,$$

$$dp/ds = p_{,\alpha}\lambda^\alpha = -\rho A_{i1}\zeta_\alpha^i \xi_1^\alpha = -\rho A_{11}.$$

Using the first of these formulas we readily find the derivative dv/ds where v is the magnitude of the velocity behind the shock line. In fact we have

$$v dv/ds = u_\alpha du_\alpha/ds = A_{i1}\zeta_\alpha^i \xi_1^\alpha = A_{21}.$$

When the flow in the region 1 is uniform the expressions for the A_{α} and the a_1 in these formulas can be taken from §4. The resulting expression for the derivative dp/ds appears to be particularly simple and is in fact given by

$$\frac{dp}{ds} = -\frac{4\rho u_n u_t \kappa}{\gamma+1} = -\frac{4\rho_1 u_{1n} u_t \kappa}{\gamma+1}.$$

6. **Curvature of stream lines behind the shock line.** The common expression for the curvature K of a differentiable curve referred to rectangular coordinates x and y is

$$K = \frac{d^2y}{dx^2} \Big/ \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}. \tag{31}$$

As so expressed the curvature is positive for a curve which is concave upwards and negative for a curve which is concave downwards. Let us put $x = x^1$ and $y = x^2$ so that the derivative $dy/dx = u^2/u^1$ in terms of the velocity components u^{α}. Then

$$\frac{d^2y}{dx^2} = \frac{\partial}{\partial x}\left(\frac{u^2}{u^1}\right) + \frac{\partial}{\partial y}\left(\frac{u^2}{u^1}\right)\frac{u^2}{u^1}. \tag{32}$$

Now define a vector w by the requirement (1) that it have the same length as the velocity vector u and (2) that the vectors u, w form a right handed system. Between the components of the vectors u and w we will then have the relations $u^1 = w^2$ and $u^2 = -w^1$. Now when we substitute the expanded form of the right member of (32) in (31) we find that the expression for K becomes

$$K = u_{\alpha,\beta} w^{\alpha} u^{\beta}/v^3, \tag{33}$$

where v is the magnitude of the velocity vector. Equation (33) gives the curvature along any stream line.

Now replace the derivatives $u_{\alpha,\beta}$ in (33) by the values given by (14). We thus obtain

$$v^3 K = A_{i\beta}\zeta^i_{\alpha}\zeta^j_{\beta} w^{\alpha}\xi^{\beta}_2 = A_{ij}\zeta^i_{\alpha} w^{\alpha}.$$

But

$$\begin{cases} \zeta^1_{\alpha} w^{\alpha} = \zeta^1_1 w^1 + \zeta^1_2 w^2 = -u_{\alpha}u_{\alpha}/u_n = -v^2/u_n, \\ \zeta^2_{\alpha} w^{\alpha} = \zeta^2_1 w^1 + \zeta^2_2 w^2 = u_{\alpha}\lambda^{\alpha}/u_n = u_t/u_n. \end{cases}$$

Hence,

$$K = -(A_{12}/vu_n) + (A_{22}u_t/v^3 u_n). \tag{34}$$

This is a general formula for the curvature of the stream lines immediately behind the shock line. For uniform flow in the region 1 formula (34) yields

$$K = \frac{-4u_t\kappa}{(\gamma+1)v} + \frac{\gamma p u_t \kappa}{v^3(\rho u_n^2 - \gamma p)}\left\{\frac{v^2[\rho]}{\rho_1} - \left(\frac{[\rho]^2}{\rho_1\rho} + \frac{8}{\gamma+1}\right)u_t^2\right\}.$$

Remark. Combining (4) and (33) we obtain the centrifugal force relation

$$p_{,\alpha}w^{\alpha} + \rho v^3 K = 0.$$

This relation holds for the plane stationary flow of a non-viscous fluid, the quantity K being the curvature of the stream line passing through the point at which the relation is considered.

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FLOW CONDITIONS NEAR THE INTERSECTION OF A SHOCK WAVE WITH SOLID BOUNDARY

By Hsue-shen Tsien

1. Introduction. Recent investigations by J. Ackeret, F. Feldmann, and N. Rott (Ref. 1) and H.-W. Liepmann (Ref. 2) on the interactions of shock and boundary layer in transonic flows have shown many novel features hitherto unrecognized. One of these new phenomena is the presence of a strong expansion zone near the solid boundary immediately behind the shock. The same situation appears also in the results of the numerical calculations of transonic flows by H. W. Emmons (Ref. 3). Since Emmons' investigation is carried out without considering the viscosity and the heat conduction of the fluid, the occurrence of expansion zone after the shock must be independent of these fluid properties. A closer consideration on the order of magnitude of quantities which enter into the phenomenon shows that the effects of the small viscosity and heat conductance of the fluid is indeed small and negligible when compared with the very large pressure and velocity changes across a shock. In other words, although the shock in transonic flow may have been caused by the presence of the boundary layer, a result of the viscosity of the fluid, the final flow characteristics can be described without introducing viscosity and heat conductance. The situation is thus not unlike that of fully developed turbulent flow which can be described without the direct aid of viscosity, although the phenomenon itself is born from the effects of viscosity.

The purpose of the present note is to show that, neglecting the effects of viscosity and heat conductance, there is a simple relation between the Mach number near the intersection of the shock with the solid boundary and the curvature of the solid boundary. This same relation was obtained by Emmons (Ref. 4) without considering the pressure gradients along the solid boundary. The present derivation is believed to be more general. The pressure gradients before and after the shock along the solid boundary are also simply related to the curvature of the boundary, with the gradient after the shock larger than the gradient before the shock.

2. Basic Equations. Let r and θ be the polar coordinates and u, v the components of velocity in the radial and circumferential direction. Further, let ρ, p be the density and pressure of the fluid. The equations of motion are then

$$\rho u \frac{\partial u}{\partial r} + \rho v \frac{\partial u}{r \partial \theta} - \frac{\rho v^2}{r} = -\frac{\partial p}{\partial r} \tag{1}$$

$$\rho u \frac{\partial v}{\partial r} + \rho v \frac{\partial v}{r \partial \theta} + \frac{\rho u v}{r} = -\frac{1}{r}\frac{\partial p}{\partial \theta} \tag{2}$$

The equation of continuity is

$$\frac{\partial \rho u}{\partial r} + \frac{\rho u}{r} + \frac{1}{r}\frac{\partial \rho u}{\partial \theta} = 0. \tag{3}$$

These equations will be applied presently to the two sides of the shock (Fig. 1). The shock is necessarily normal to the boundary as the boundary is considered to have continuous slope and the flow immediately to the boundary should not change direction by passing through the shock. As a further simplification of the problem the shock will be assumed to be straight. Since the following calculation is limited to the immediate neighborhood of the point P of intersection of the shock and the solid boundary, the effect of any finite curvature of the shock will be of second order. Therefore the assumption of a straight shock really does not restrict the physical problem considered. Let the subscript 1 and 2 denote quantities corresponding to the upstream side and to the dowstream side respectively, and let the superscript 0 denote quantities at

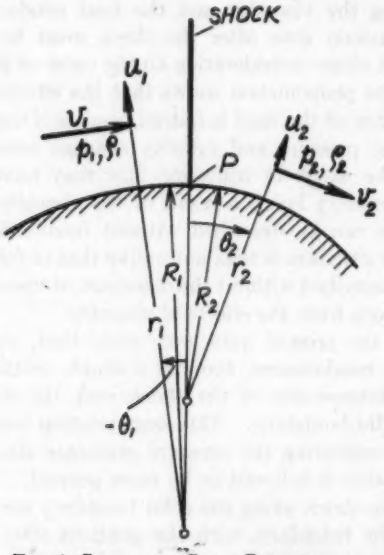

FIG. 1. SHOCK AND SOLID BOUNDARY

point P. Furthermore, let R_1 and R_2 be the radii of curvature of the solid boundary at left and at right of P. The investigation then allows for a change in the curvature of the boundary, even though the slope is continuous. Since the radial component of velocity is zero at the solid boundary, equation (1) gives

$$\frac{\rho_1^0 v_1^{0^2}}{R_1} = \left(\frac{\partial p_1}{\partial r_1}\right)^0.$$

Since $\gamma p_1^0/\rho_1^0 = a_1^{0^2}$, γ being the ratio of specific heat, and $v_1^0/a_1^0 = M_1^0$ which is the Mach number of flow at the left of the point P, the above equation can be written as

$$\left[\frac{\partial(p_1/p_1^0)}{\partial r_1}\right]^0 = \frac{1}{R_1}\gamma M_1^{0^2}. \tag{4}$$

Similarly,

$$\left[\frac{\partial(p_2/p_2^0)}{\partial r_2}\right]^0 = \frac{1}{R_2}\gamma M_2^{0^2}. \tag{5}$$

By specializing equation (2) for the left of point P, one has

$$-\frac{1}{R_1}\left(\frac{\partial p_1}{\partial \theta_1}\right)^0 = \rho_1^0 v_1^0\left(\frac{\partial v_1}{r_1 \partial_1 \theta}\right)^0 = v_1\left(\frac{\partial \rho_1 v_1}{r_1 \partial \theta_1}\right)^0 - v_1^{0^2}\left(\frac{\partial \rho_1}{r_1 \partial \theta_1}\right)^0. \tag{6}$$

But

$$\left(\frac{\partial \rho_1}{r_1 \partial \theta_1}\right)^0 = \left(\frac{d\rho_1}{dp_1}\right)^0\left(\frac{\partial p_1}{r_1 \partial \theta_1}\right) = \frac{1}{a^{0^2}}\left(\frac{\partial p_1}{r_1 \partial \theta_1}\right)^0.$$

Here $(dp_1/d\rho_1)^0$ is identified with $a_1^{0^2}$ because along any stream line in a non-viscous flow the fluid undergoes isentropic compression or expansion. Equation (3) gives

$$\left(\frac{\partial \rho_1 v_1}{r_1 \partial \theta_1}\right)^0 = -\left(\frac{\partial \rho_1 u_1}{\partial r_1}\right)^0 = -\rho_1^0\left(\frac{\partial u_1}{\partial r_1}\right)^0.$$

Therefore, equation (6) can be written as

$$(M_1^{0^2} - 1)\frac{1}{R_1}\left(\frac{\partial p_1}{\partial \theta_1}\right)^0 = -\rho_1^0 v_1^0\left(\frac{\partial u_1}{\partial r_1}\right)^0. \tag{7}$$

Similarly for the right of the point P, one has

$$(1 - M_2^{0^2})\frac{1}{R_2}\left(\frac{\partial p_2}{\partial \theta_2}\right)^0 = \rho_2^0 v_2^0\left(\frac{\partial u_2}{\partial r_2}\right)^0. \tag{8}$$

3. Relations of Quantities on the Two Sides of Shock. Now introduce the independent variable η as the normal distance from the solid boundary, i.e.,

$$\eta = r_1 - R_1 = r_2 - R_2 \tag{9}$$

Then equation (4) gives the pressure p_1 along the left of the shock:

$$p_1/p_1^0 = 1 + \gamma M_1^{0^2}\eta/R_1 + \cdots \tag{10}$$

Now the flow ahead of the shock can generally be considered as irrotational. Then the flow is not only isentropic along a given stream line but has the same entropy throughout the field. The temperature ratio T_1/T_1^0 is then given by

$$T_1/T_0^1 = (p_1/p_1^0)^{(\gamma-1)/\gamma} = 1 + (\gamma - 1)M_1^{0^2}\eta/R_1 + \cdots \tag{11}$$

However,

$$T_1/T_1^0 = \left(1 + \frac{\gamma - 1}{2}M_1^{0^2}\right)\Big/\left(1 + \frac{\gamma - 1}{2}M_1^2\right). \tag{12}$$

Equations (11) and (12) then give the Mach number M_1 as follows:

$$M_1^2 = M_1^{0^2}\left[1 - 2\left(1 + \frac{\gamma - 1}{2}M_1^{0^2}\right)\frac{\eta}{R_1} + \cdots\right]. \tag{13}$$

This is the Mach number along the left of the shock.

The incident velocity to the shock is not normal for points away from the solid boundary because of the presence of radial velocity u_1 at such points. However, a moment's reflection will show that this effect is of second order in calculating pressure p_2. Hence, for the accuracy adopted, the normal shock formula can be used. Equations (10) and (13) then give the pressure ratio as

$$\frac{p_2}{p_1^0} = \frac{p_1}{p_1^0}\left(\frac{2\gamma}{\gamma+1}M_1^2 - \frac{\gamma-1}{\gamma+1}\right)$$

$$= \left(\frac{2\gamma}{\gamma+1}M_1^{0^2} - \frac{\gamma-1}{\gamma+1}\right)\left[1 + \frac{2\gamma M_1^{0^4} - \gamma(\gamma+3)M_1^{0^2}}{2\gamma M_1^{0^2} - (\gamma-1)}\frac{\eta}{R_1} + \cdots\right].$$

Therefore,

$$\frac{p_2}{p_2^0} = 1 + \frac{2\gamma M_1^{0^4} - \gamma(\gamma+3)M_1^{0^2}}{2\gamma M_1^{0^2} - (\gamma-1)}\frac{\eta}{R_1} + \cdots. \tag{14}$$

On the other hand, equation (5) requires that

$$\left[\frac{\partial(p_2/p_2^0)}{\partial\eta}\right]^0 = \frac{1}{R_2}\frac{2\gamma + \gamma(\gamma-1)M_1^{0^2}}{2\gamma M_1^{0^2} - (\gamma-1)}. \tag{15}$$

By substituting equation (14) into equation (15), one has

$$[2 + (\gamma-1)M_1^{0^2}]R_1/R_2 = 2M_1^{0^4} - (\gamma+3)M_1^{0^2}$$

or,

$$M_1^{0^4} - \frac{1}{2}\left\{(\gamma+3) + \frac{R_1}{R_2}(\gamma-1)\right\}M_1^{0^2} - \frac{R_1}{R_2} = 0. \tag{16}$$

The appropriate root of this quadratic equation is

$$M_1^{0^2} = \frac{1}{4}\left\{(\gamma+3) + \frac{R_1}{R_2}(\gamma-1)\right\} + \sqrt{\frac{1}{16}\left\{(\gamma+3) + \frac{R_1}{R_2}(\gamma-1)\right\}^2 + \frac{R_1}{R_2}}. \tag{17}$$

Hence, the Mach numbers before and after the shock are not free but definitely determined by the ratio of the radii of curvature of the solid boundary. This is the same relation obtained by Emmons (Ref. 4) without considering the pressure gradients along the solid boundary. Fig. 2 shows this relation by assuming $\gamma = 1.4$. It is seen that for large values of Mach number before the shock, the curvature of the boundary after the shock must be larger than that before the shock. If the curvature of the boundary is the same, then $M_1^0 = 1.661$.

The radial component of velocity u is not altered by going through the shock. Thus $u_1 = u_2$ along the shock. Furthermore, the condition of continuity requires that $\rho_1^0 v_1^0 = \rho_2^0 v_2^0$. Therefore, equations (7) and (8) yield immediately the relation

$$\left(\frac{\partial p_2}{\partial s}\right)^0 \Big/ \left(\frac{\partial p_1}{\partial s}\right)^0 = -\frac{M_1^{0^2} - 1}{1 - M_1^{0^4}}, \tag{18}$$

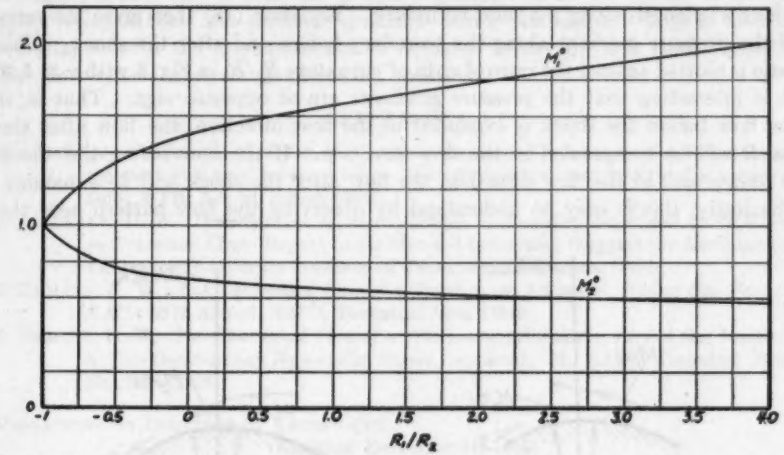

FIG. 2. Relation between the Mach number M_1° before the Shock, the Mach number M_2° after the Shock at the Boundary and the Ratio R_1/R_2 of the Radii of Curvature of the Boundary.

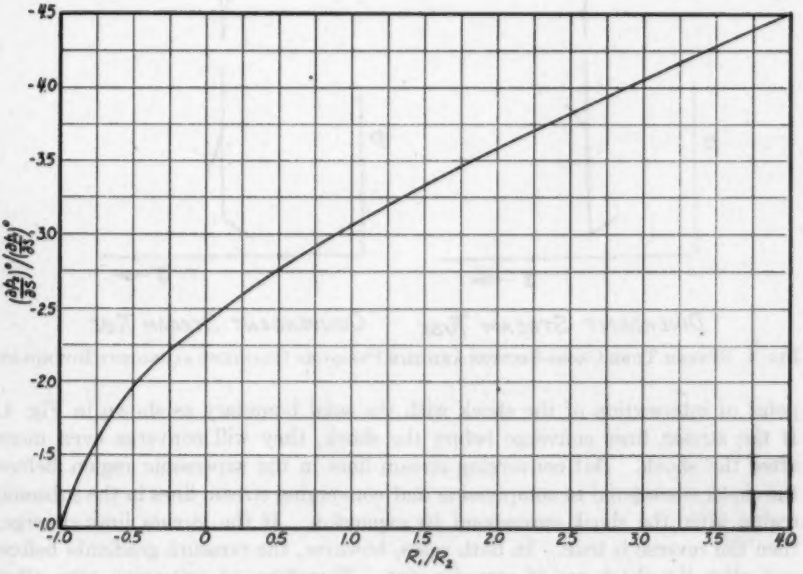

FIG. 3. Ratio $\left(\frac{\partial p_2}{\partial s}\right)^\circ \Big/ \left(\frac{\partial p_1}{\partial s}\right)^\circ$ of the Pressure Gradients along the Boundary after the Shock and before the Shock as Function of the Ratio R_1/R_2 of the Radii of Curvature of the Boundary.

where s is length along the solid boundary. Equation (18) then gives the ratio of the pressure gradient along the boundary before and after the shock. This ratio is plotted against the ratio of radii of curvature R_1/R_2 in Fig. 3 with $\gamma = 1.4$. It is interesting that the pressure gradients are of opposite sign. That is, if the flow before the shock is expanded in the flow direction, the flow after the shock will be compressed in the flow direction. If the flow before the shock is compressed in the flow direction, the flow after the shock will be expanded. Physically, this is easy to understand by observing the flow pattern near the

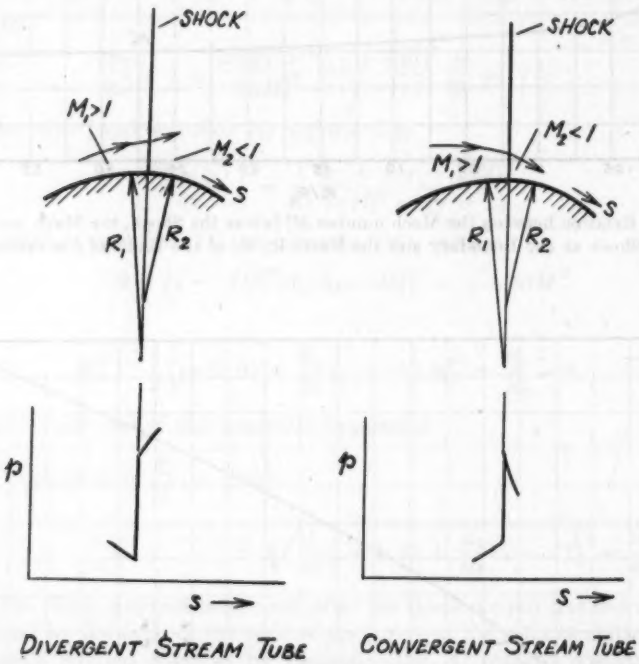

FIG. 4. STREAM TUBE CROSS-SECTION AND THE PRESSURE GRADIENT ALONG THE BOUNDARY

point of intersection of the shock with the solid boundary as shown in Fig. 4. If the stream lines converge before the shock, they will converge even more after the shock. But converging stream lines in the supersonic region before the shock correspond to compression and converging stream lines in the subsonic region after the shock correspond to expansion. If the stream lines diverge, then the reverse is true. In both cases, however, the pressure gradients before and after the shock are of opposite sign. Therefore, an expansion zone after the shock is necessarily connected with a compression zone before the shock. This is in complete agreement with the experimental data (Ref. 1) and Emmons' calculations (Ref. 3).

Figure 3 also shows that the pressure gradient after the shock is always much steeper than the pressure gradient before the shock. This is also in accordance with the measurements of Ackeret, Feldmann and Rott (Ref. 1). The flow after the shock is, of course, rotational.

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(Received December 12, 1946)

CORRECTIONS ON THE PAPER "ONE DIMENSIONAL FLOWS OF A GAS CHARACTERIZED BY VAN DER WAAL'S EQUATION OF STATE"

By Hsue-shen Tsien

In the recent paper by the author,[1] a mistake[2] has been made in not including the variation of internal energy of the gas caused by change in volume. As a result, the formulae are incorrect in the terms connected with the parameter a of the Van der Waal's equation. The following corrected equations should be substituted for the corresponding equations in that paper.

$$C_v dT + T(\partial p/\partial T)d\rho^{-1} = 0 \tag{2}$$

$$C_v dT - \{RT(\rho - b\rho^2)^{-1}\}d\rho = 0 \tag{3}$$

$$(\alpha + \beta T + \gamma T^2)dT - \{RT(\rho - b\rho^2)^{-1}\}d\rho = 0 \tag{5}$$

$$dT^{(1)}(\rho) = 0\ (8), \qquad T^{(1)}(\rho) = 0 \tag{13}$$

$$T = C\rho^{R/\alpha} + b\frac{CR}{\alpha}\rho^{R/\alpha+1} - \beta\frac{C^2}{\alpha}\rho^{2R/\alpha} - \gamma\frac{C^3}{2\alpha}\rho^{3R/\alpha} \tag{17}$$

and similar correction for (18). The term involving $(a\rho_0/RT_0)$ in (20) and (21) should be dropped. For (22), (23), (24), (25), (31), (32), and (34) the factor in connection with the parameter a should be corrected according to the following scheme:

Equation	Incorrect Factor	Correct Factor
(22)	$\frac{1}{1-\lambda}$	1
(23)	$-\frac{1}{1-\lambda^2}$	$-\frac{1}{1+\lambda}$
(24)	$-\frac{\lambda}{1-\lambda}$	1
(25)	$\frac{\lambda^2}{1-\lambda^2}$	$-\frac{\lambda}{1+\lambda}$
(31)	$\frac{1}{1-\lambda}$	1
(32)	$-\frac{1}{1+\lambda}$	$-\frac{1-\lambda}{1+\lambda}$
(34)	$-\frac{1}{1+\lambda}$	$-\frac{1-\lambda}{1+\lambda}$

[1] Journal of Mathematics and Physics, *25*, pp. 301–324 (1947)

[2] The author is indebted to Professor Joseph H. Keenan for calling his attention to this mistake.

In addition, the correct form of other eqtations are:

$$C_v dT + d(p/\rho) - a d\rho - dp/\rho = 0 \tag{27}$$

$$(\alpha + \beta T + \gamma T^2) dT + d(p/\rho) - a d\rho + v dv = 0 \tag{28}$$

$$\tfrac{1}{2}v^2 = \alpha T_0\left\{1 - \frac{T}{T_0}\right\} + \frac{\beta T_0^2}{2}\left\{1 - \frac{T^2}{T_0^2}\right\} + \frac{\gamma T_0^3}{3}\left\{1 - \frac{T^3}{T_0^3}\right\} + \frac{p_0}{\rho_0}\left\{1 - \left(\frac{p}{p_0}\right)\left(\frac{\rho_0}{\rho}\right)\right\} - a\rho_0\left(1 - \frac{\rho}{\rho_0}\right) \tag{29}$$

$$t_1 = \frac{\lambda}{1+\lambda}\left\{(M^2+2)\left(1 + \frac{\lambda}{2}M^2\right)^{-1/\lambda} - 2\right\} \tag{36}$$

$$P_1 = (M^2+2)\left(1 + \frac{\lambda}{2}M^2\right)^{-1/\lambda} - 1 + \left(1 + \frac{\lambda}{2}M^2\right)^{1-1/\lambda} \tag{39}$$

$$\gamma_1 = \frac{1}{1+\lambda}\left\{(M^2+2)\left(1 + \frac{\lambda}{2}M^2\right)^{-1/\lambda} - 2\right\} \tag{40}$$

$$f_1 = \frac{1}{1+\lambda}\left[(2+\lambda)\left(1 + \frac{\lambda}{2}\right)^{-1/\lambda} - \left\{(1+\lambda) + M^2\right\}\left(1 + \frac{\lambda}{2}M^2\right)^{-1/\lambda}\right] \tag{46}$$

$$-a\rho_1 + \alpha T_1 + \frac{\beta}{2}T_1^2 + \frac{\gamma}{3}T_1^3 + \frac{p_1}{\rho_1} + \frac{1}{2}v_1^2 = -a\rho_2 + \alpha T_2 + \frac{\beta}{2}T_2^2 + \frac{\gamma}{3}T_2^3 + \frac{p_2}{\rho_2} + \frac{1}{2}v_2^2 \tag{49}$$

Π_1, as given by (63) should be multiplied by $(1 - \lambda)$, and the second term for D_1 in (65) should be also multiplied by $(1 - \lambda)$. The corrected expressions for m_1, m_2, m_3, m_4 in (69) are:

$$\left.\begin{aligned}
m_1 &= \frac{1-\lambda}{2(1+\lambda)}\left[\frac{(2+\lambda)^3 M_1^4}{(\lambda M_1^2+2)^2\{2(1+\lambda)M_1^2 - \lambda\}} - 1\right] - \frac{1}{2}(D_1 + \Pi_1)\\
m_2 &= \frac{1}{2}\left[\frac{(2+\lambda)M_1^2}{\lambda M_1^2 + 2} - 1\right] - \frac{1}{2}(D_2 + \Pi_2)\\
m_3 &= \frac{\lambda}{2(1+\lambda)}\left[\frac{\{2(1+\lambda)M_1^2 - \lambda\}(\lambda M_1^2+2)}{(2+\lambda)^2 M_1^2} - 1\right] - \frac{1}{2}(D_3 + \Pi_3)\\
m_4 &= \frac{\lambda}{2(1+\lambda)}\left[\frac{\{2(1+\lambda)M_1^2 - \lambda\}^2(\lambda M_1^2+2)^2}{(2+\lambda)^4 M_1^4} - 1\right] - \frac{1}{2}(D_4 + \Pi_4)
\end{aligned}\right\} \tag{69}$$

The tabulated functions are correct except t_1, P_1, r_1, f_1, Π_1, D_1, τ_1, and the m's.

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CORRECTIONS TO PAPER "MATHIEU FUNCTIONS AND THEIR CLASSIFICATION"

By N. W. McLachlan

In this paper which appeared in volume *25*, p. 209, 1946 the following corrections should be made:

p. 211 In (26), (27); for S read s

213 2nd line below; (12) for z^{2m-2} read 2^{2m-2}

219 In (15); for cos read sin; delete the last sentence

230 3rd line below (8) §23; for *ce* read *ceu*

236 2nd line above §31; for $\frac{1}{4}m$ read $\frac{1}{4}\pi i$

239 In (12)–(15); for c, s read C, S; in (15) change the second $-$ to $+$

240 In (17); multiply the denominator by $r!$, and in (22) insert n at the r.h.s. 2nd line below (22); delete $\cdots Gek_{2n}$.

ON SPHEROIDAL WAVE FUNCTIONS OF ORDER ZERO

By C. J. Bouwkamp

1. Introduction. Methods have been developed for obtaining the characteristic values of linear second-order differential equations such as Mathieu's equation and that of spheroidal wave functions.[1]

In a recent paper Miss Blanch (5) has given a method for correcting the characteristic values of Mathieu's equation, and the coefficients, occurring in the Fourier series developments of the characteristic functions. According to Miss Blanch "there does not seem to appear in the literature any method for improving the accuracy of the characteristic values, except by cumbersome iteration."

We would like to draw attention to the fact, however, that in our thesis (6) exactly the same method as described by Miss Blanch, was developed for spheroidal wave functions. As is emphasized by Miss Blanch, the method in question is actually applicable to all types of equations whenever the coefficients, occurring in the development of the characteristic function, are determined by a three-term recursion formula.

As our thesis is published in Dutch, and during the war, it is easily understood why our method was unavailable abroad. Therefore it may be worth while to give here a survey of some of the results obtained previously by the present author. In addition some new information will be given.

2. Spheroidial wave functions of order zero. The differential equation for spheroidal wave functions of order zero is

$$d\{(1 - \xi^2)dX/d\xi\}/d\xi + (k^2\xi^2 + \Lambda)X = 0 \tag{1}$$

wherein k is given parameter and Λ one of the countable set of characteristic values $\Lambda_0, \Lambda_1, \Lambda_2, \cdots$, all depending on k, such that for $\Lambda = \Lambda_m$, (1) has one and only one solution $X_m(\xi)$ that is an integral function of ξ.

Equation (1) can be solved by a development in Legendre polynomials:

$$X(\xi) = \sum_0^\infty b_n P_n(\xi) \tag{2}$$

When (2) is substituted in (1), the following three-term recursion formula for the coefficients b_n is obtained:

$$k^2 \frac{(n+1)(n+2)}{(2n+3)(2n+5)} b_{n+2} + \left\{\Lambda - n(n+1) + \frac{2n(n+1)-1}{(2n-1)(2n+3)} k^2\right\} b_n + k^2 \frac{n(n-1)}{(2n-1)(2n-3)} b_{n-2} = 0 \tag{3}$$

[1] Cf. Strutt, M. J. O., Lamésche- Mathieusche- und verwandte Funktionen in Physik und Technik; Berlin, 1932.

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The required solution b_n of this difference equation must satisfy two different and independent conditions, namely i) $b_n \to 0$ for $n \to \infty$; ii) $b_n = 0$ for $n < 0$.

Upon introducing the abbreviations

$$\alpha_n = n(n+1) - \frac{k^2}{2}\left\{1 + \frac{1}{(2n-1)(2n+3)}\right\} \qquad (n \geq 0) \tag{4}$$

$$\beta_n = \frac{n^2(n-1)^2}{(2n-1)^2}\,\frac{k^4}{(2n-3)(2n+1)} \qquad (n \geq 2) \tag{5}$$

$$N_n = k^2\,\frac{n(n-1)}{(2n-1)(2n+1)}\,\frac{b_n}{b_{n-2}} \qquad (n \geq 2) \tag{6}$$

we get from (3)

$$N_n = \beta_n/(\alpha_n - \Lambda - N_{n+2}) \qquad (n \geq 2) \tag{7}$$

$$N_{n+2} = \alpha_n - \Lambda - (\beta_n/N_n) \qquad (n \geq 2) \tag{8}$$

Condition i) will be satisfied if we take, by iteration of (7),

$$N_n = \frac{\beta_n}{|\,\alpha_n - \Lambda} - \frac{\beta_{n+2}}{|\,\alpha_{n+2} - \Lambda} - \frac{\beta_{n+4}}{|\,\alpha_{n+4} - \Lambda} - \cdots \tag{9}$$

This infinite continued fraction converges, given k, Λ, if n is large enough, to a non-zero, finite value.

On account of condition ii), by iteration of (8) one has

$$N_{n+2} = \alpha_n - \Lambda - \frac{\beta_n}{|\,\alpha_{n-2} - \Lambda} - \frac{\beta_{n-2}}{|\,\alpha_{n-4} - \Lambda} - \cdots \tag{10}$$

The last denominator is $\alpha_0 - \Lambda$ or $\alpha_1 - \Lambda$, depending on whether n is even or odd. Strictly speaking, certain values of Λ should be excluded in (10), because N_{n+2} becomes infinitely large if N_n happens to be zero.

Let us now take $n = m + 2$ in (9) and $n = m$ in (10). Then (9) and (10) must give the same result. Therefore, the eigenvalues of (1) are the roots of the following transcendental equation

$$U(\Lambda) = U_1(\Lambda) + U_2(\Lambda) = 0 \tag{11}$$

wherein

$$U_1(\Lambda) = \alpha_m - \Lambda - \frac{\beta_m}{|\,\alpha_{m-2} - \Lambda} - \frac{\beta_{m-2}}{|\,\alpha_{m-4} - \Lambda} - \cdots \tag{12}$$

$$U_2(\Lambda) = -\frac{\beta_{m+2}}{|\,\alpha_{m+2} - \Lambda} - \frac{\beta_{m+4}}{|\,\alpha_{m+4} - \Lambda} - \frac{\beta_{m+6}}{|\,\alpha_{m+6} - \Lambda} - \cdots \tag{13}$$

Until now m is wholly arbitrary. As

$$\Lambda_m\,,\ \alpha_m \to m(m+1) \qquad \text{for } k \to 0,$$

it will be obvious that, at least for small values of k, it is just the m^{th} eigenvalue Λ_m which can most easily be computed from (11), (12), (13). For large values

of k, however, m may be better taken greater than the index of the required characteristic value. In other words, if $b_{m'}$ is the largest coefficient of all b_n occurring in (2), then in (12), (13) m should be taken equal to m'.

It may be noticed that N_n plays the same rôle as G_n in Miss Blanch's analysis. Furthermore her α, θ are the analogues of our Λ, k, respectively. The only difference is that in the problem of spheroidal functions more intricate coefficients α_n, β_n (analogous to k_n in Miss Blanch's analysis) occur.

The main part of the above analysis is well known and has been often treated in the literature. For further details the reader may be referred to the list of references at the end of this paper.

3. Method of approximation. Let λ_m be some approximation of the m^{th} eigenvalue Λ_m. If λ_m would be the correct value of Λ_m, then $U(\lambda_m)$ in (11) would be equal to zero. The less the absolute value of $U(\lambda_m)$, the better the approximation. Therefore it is clear that, from the very value of $U(\lambda_m)$, it must be possible to derive a correction $\delta\lambda$ such that $\lambda_c = \lambda_m + \delta\lambda$ is a better approximation of Λ_m than is the initial λ_m.

Now, from (8) $\delta N_{n+2} = -\delta\lambda + (\beta_n/N_n^2)\cdot\delta N_n$. Hence, by iteration

$$\delta U_1 = -\delta\lambda\left(1 + \frac{\beta_m}{N_m^2} + \frac{\beta_m\beta_{m-2}}{N_m^2 N_{m-2}^2} + \frac{\beta_m\beta_{m-2}\beta_{m-4}}{N_m^2 N_{m-2}^2 N_{m-4}^2} + \cdots\right)$$

Similarly, from (7)

$$\delta U_2 = -\delta\lambda\left(\frac{N_{m+2}^2}{\beta_{m+2}} + \frac{N_{m+2}^2 N_{m+4}^2}{\beta_{m+2}\beta_{m+4}} + \frac{N_{m+2}^2 N_{m+4}^2 N_{m+6}^2}{\beta_{m+2}\beta_{m+4}\beta_{m+6}} + \cdots\right)$$

Furthermore

$$0 = U(\Lambda_m) = U(\lambda_m) + \delta U_1 + \delta U_2,$$

and consequently, for the correction

$$\delta\lambda = \frac{U_1(\lambda_m) + U_2(\lambda_m)}{\left(1 + \frac{\beta_m}{N_m^2} + \frac{\beta_m\beta_{m-2}}{N_m^2 N_{m-2}^2} + \cdots\right) + \left(\frac{N_{m+2}^2}{\beta_{m+2}} + \frac{N_{m+2}^2 N_{m+4}^2}{\beta_{m+2}\beta_{m+4}} + \cdots\right)} \tag{14}$$

Let us first demonstrate this method of approximation for $m = 3$ and $k^2 = 6$. The coefficients α_n, β_n, can be calculated from (4), (5), respectively.

$$\alpha_1 = -1.600000$$

α_3	= 8.933333	β_3	= 2.468571
α_5	= 26.97436	β_5	= 2.308802
α_7	= 52.9864	β_7	= 2.277353
α_9	= 86.992	β_9	= 2.26582
α_{11}	= 128.99	β_{11}	= 2.2603
α_{13}	= 179.0	β_{13}	= 2.257
α_{15}	= 237	β_{15}	= 2.26

An approximate value of Λ_3 is $\lambda_3 = 9.04$. The infinite continued fraction (13) may be broken up at $N_{17} = 0$; then from (7) consecutively

$$N_{15} = 0.01,\ N_{13} = 0.013,\ N_{11} = 0.0188,$$

$$N_9 = 0.02907,\ N_7 = 0.051855,\ N_5 = 0.129110 = -U_2(\lambda_3)$$

Similarly, from the other side, using (8), (12)

$$N_3 = -10.640000,\ N_5 = 0.125342 = U_1(\lambda_3)$$

Consequently

$$U_1(\lambda_3) + U_2(\lambda_3) = -0.003768$$

Furthermore, applying the above values of the N's,

$$\left(1 + \frac{\beta_3}{N_3^2}\right) + \left(\frac{N_5^2}{\beta_5} + \frac{N_5^2 N_7^2}{\beta_5 \beta_7} + \cdots\right)$$

$= (1 + 0.021805) + (0.007200 + 0.000008) = 1.029013$. For the correction finally

$$\delta\lambda = -0.003768/1.029013 = -0.003662$$

The corrected characteristic value is thus $\lambda_c = 9.036338$. The corrected values of the N's are

$$\text{"downwards": } N_7 = 0.051851;\ N_5 = 0.129083,$$

$$\text{"upwards": } N_3 = -10.636338;\ N_5 = 0.129083,$$

showing that N_5 is the same, whether computed from the finite or the infinite continued fraction. *Thus our method at once leads from two-decimal accuracy to a six-decimal one.*

This is an improvement on the cumbersome iteration method in earlier work on Mathieu's and similar equations. The method is fully equivalent to that of Miss Blanch.

From the corrected N_n, one can easily obtain the ratios of successive coefficients b_n; see (6). It is convenient to normalize them, such that, independently of k,

$$\int_{-1}^{+1} X_m^2(\xi)\, d\xi = \int_{-1}^{+1} P_m^2(\xi)\, d\xi = 2/(2m+1) \tag{15}$$

with the further condition $b_m \to 1$ for $k \to 0$.

In current literature, the coefficients b_n are often normalized such that $b_m = 1$ for the m^{th} characteristic function. This, however, sometimes leads to a serious difficulty, as for some values of k the m^{th} eigenfunction does not contain a term P_m at all. Only in case of our normalization (13) are all coefficients b_n finite throughout. The analogue in the theory of Mathieu's equation is well known.

4. Numerical calculations for small values of k^2. For small values of k^2, the characteristic values Λ_m and the corresponding coefficients b_n can be computed from series in powers of k^2. Although these developments certainly have non-vanishing circles of convergence, their actual radii of convergence are not known. As is observed from practical numerical calculations, the range of convergence seems to increase with increasing m.

Substituting $\Lambda = a + bk^2 + ck^4 + \cdots$ in (12), (13), we can develop U_1 and U_2 into powers of k^2. Then, by application of (11), the unknown coefficients $a, b, c, \cdots$, can be calculated successively. In the same manner, by (6), (7), (8) and (15), the normalized b_n can be expanded into powers of k^2.

We have found for $m = 0$

$$\Lambda_0 = -\frac{1}{3}k^2 - \frac{2}{3^3\cdot 5}k^4 - \frac{2^2}{3^5\cdot 5\cdot 7}k^6 + \frac{2\cdot 13}{3^7\cdot 5^2\cdot 7}k^8 + \frac{2\cdot 23}{3^9\cdot 5^2\cdot 7\cdot 11}k^{10} + \frac{2^2\cdot 19\cdot 6763}{3^{10}\cdot 5^3\cdot 7^2\cdot 11\cdot 13}k^{12} + \cdots$$

$$= -\frac{1}{3}k^2 - \frac{2}{135}k^4 - \frac{4}{8505}k^6 + \frac{26}{1913625}k^8 + \frac{92}{37889775}k^{10} + \frac{513988}{9050920003125}k^{12} + \cdots$$

$$b_0 = 1 - \frac{1}{2\cdot 3^4\cdot 5}k^4 - \frac{2}{3^6\cdot 5\cdot 7}k^6 + \frac{713}{2^3\cdot 3^8\cdot 5^4\cdot 7}k^8 + \cdots$$

$$b_2 = \tfrac{1}{5}k^2 + \frac{2}{3^4\cdot 7}k^4 - \frac{61}{2\cdot 3^6\cdot 5^2\cdot 7}k^6 - \frac{79}{3^8\cdot 5\cdot 7\cdot 11}k^8 + \cdots$$

$$b_4 = \frac{1}{3\cdot 5^2\cdot 7}k^4 + \frac{4}{3^5\cdot 5^2\cdot 7\cdot 11}k^6 - \frac{10001}{2\cdot 3^5\cdot 5^4\cdot 7^2\cdot 11\cdot 13}k^8 + \cdots$$

$$b_6 = \frac{1}{3^3\cdot 5\cdot 7^2\cdot 11}k^6 + \frac{2}{3^5\cdot 5^2\cdot 7^2\cdot 11}k^8 + \cdots$$

$$b_8 = \frac{1}{3^6\cdot 5^2\cdot 7\cdot 11\cdot 13}k^8 + \cdots$$

$$b_{2n+1} = 0$$

$$X_0(1) = 1 + \tfrac{1}{5}k^2 + \frac{17}{2\cdot 3^4\cdot 5^2}k^4 - \frac{1}{2\cdot 3^2\cdot 5\cdot 7^2}k^6 - \frac{17\cdot 29}{2^3\cdot 3^7\cdot 5^3\cdot 7}k^8 + \cdots$$

$$X_0(0) = 1 - 1\tfrac{1}{5}k^2 - \frac{37}{2^3\cdot 3^4\cdot 5^2}k^4 + \frac{313}{2^4\cdot 3^5\cdot 5^2\cdot 7^2}k^6 + \frac{28813}{2^7\cdot 3^7\cdot 5^3\cdot 7^2}k^8 + \cdots$$

It may be remarked that also Stier (24) obtained the formula for Λ_0 as given above, apart from a printer's error (in the denominator of the term with k^{12}, 5^3 instead of the correct 5^5). The last term (being k^8) of Niven's (19) as well as of Wilson's (30) formula is incorrect ($182/3^7\cdot 5^3\cdot 7^3 = 26/3^7\cdot 5^3\cdot 7^2$ and $16/3^7\cdot 5^3\cdot 7^2$, respectively, instead of the correct $26/3^7\cdot 5^5\cdot 7$).

In our own previous result (6) the term with k^{10} was incorrect too.

When $m = 1$ we have

$$\Lambda_1 = 2 - \frac{3}{5}k^2 - \frac{2\cdot 3}{5^3\cdot 7}k^4 + \frac{2^2}{3\cdot 5^5\cdot 7}k^6 + \frac{2\cdot 229}{3\cdot 5^6\cdot 7^3\cdot 11}k^8 - \frac{2^2\cdot 6271}{3\cdot 5^9\cdot 7^3\cdot 11\cdot 13}k^{10} - \frac{2^2\cdot 11108467}{3^5\cdot 5^{11}\cdot 7^5\cdot 11\cdot 13}k^{12} + \cdots$$

$$= 2 - \frac{3}{5}k^2 - \frac{6}{875}k^4 + \frac{4}{65625}k^6 + \frac{458}{176859375}k^8 - \frac{25084}{287396\ 484375}k^{10} - \frac{44\ 433868}{28516\ 916162\ 109375}k^{12} + \cdots$$

$$b_1 = 1 - \frac{3}{2\cdot 5^4\cdot 7}k^4 + \frac{2}{3\cdot 5^5\cdot 7}k^6 + \frac{8557}{2^3\cdot 3\cdot 5^8\cdot 7^4}k^8 + \cdots$$

$$b_3 = \frac{1}{5^2}k^2 - \frac{2}{3^2\cdot 5^4}k^4 - \frac{4369}{2\cdot 3^2\cdot 5^6\cdot 7^2\cdot 11}k^6 + \frac{21551}{3^2\cdot 5^8\cdot 7^2\cdot 11\cdot 13}k^8 + \cdots$$

$$b_5 = \frac{1}{3^2\cdot 5\cdot 7^2}k^4 - \frac{4}{3^2\cdot 5^3\cdot 7^2\cdot 13}k^6 - \frac{6317}{2\cdot 3^4\cdot 5^5\cdot 7^4\cdot 13}k^8 + \cdots$$

$$b_7 = \frac{1}{3^4\cdot 5\cdot 7\cdot 11\cdot 13}k^6 - \frac{2}{3^3\cdot 5^3\cdot 7\cdot 11\cdot 13\cdot 17}k^8 + \cdots$$

$$b_9 = \frac{1}{3^3\cdot 5^2\cdot 7\cdot 11^2\cdot 13\cdot 17}k^8 + \cdots$$

$$b_{2n} = 0$$

$$X_1(1) = 1 + \frac{1}{5^2}k^2 - \frac{3}{2\cdot 5^3\cdot 7^2}k^4 - \frac{641}{2\cdot 3^4\cdot 5^5\cdot 7^2}k^6 + \frac{12182647}{2^3\cdot 3^4\cdot 5^7\cdot 7^4\cdot 11^2}k^8 + \cdots$$

$$\frac{d}{d\xi}X_1(0) = 1 - \frac{3}{2\cdot 5^2}k^2 + \frac{51}{2^3\cdot 5^3\cdot 7^2}k^4 + \frac{6641}{2^4\cdot 3^4\cdot 5^5\cdot 7^2}k^6 - \frac{41469889}{2^7\cdot 3^2\cdot 5^7\cdot 7^4\cdot 11^2}k^8 + \cdots$$

Stier (23) has given Λ_1 up to k^{10}. His last coefficient (of k^{10}) has only one factor 7 in the denominator instead of three. There is obviously also a printer's error in his k^6 term.

We have good reasons to believe that the formulas above are correct throughout, as during the careful calculations many checks were applied.

The larger m, the more intricate the numerical constants in the power series. In addition to the formulas already given for $m = 0$, 1, we have for $m = 2$:

$$\Lambda_2 = 6 - \frac{11}{21}k^2 + \frac{94}{9261}k^4 + \frac{21388}{44925111}k^6 - \frac{3633830}{257555\,661363}k^8 - \frac{47862\,87764}{1963\,346806\,570149}k^{10} + \cdots$$

$$b_0 = -\frac{1}{3^2\cdot 5}k^2 - \frac{2}{3^4\cdot 5\cdot 7}k^4 + \frac{811}{2\cdot 3^6\cdot 5\cdot 7^4}k^6 + \cdots$$

$$b_2 = 1 - \frac{5\cdot 109}{2\cdot 3^4\cdot 7^4}k^4 - \frac{2\cdot 36781}{3^6\cdot 7^6\cdot 11}k^6 + \cdots$$

$$b_4 = \frac{2\cdot 3}{5\cdot 7^2}k^2 - \frac{2^2}{5\cdot 7^4\cdot 11}k^4 - \frac{2^2\cdot 29\cdot 263}{3^5\cdot 5\cdot 7^6\cdot 11\cdot 13}k^6 + \cdots$$

$$b_6 = \frac{1}{3^2\cdot 7^2\cdot 11}k^4 - \frac{2^2}{3^3\cdot 7^4\cdot 11\cdot 13}k^6 + \cdots$$

$$b_8 = \frac{2^2}{3^4\cdot 5\cdot 7\cdot 11^2\cdot 13}k^6 + \cdots$$

$$b_{2s+1} = 0$$

For $m = 3$:

$$\Lambda_3 = 12 - \frac{23}{45}k^2 + \frac{3302}{1002375}k^4 - \frac{1572668}{26387\,521875}k^6 - \frac{168179\,303026}{63362\,904964\,734375}k^8 + \cdots,$$

for $m = 4$:

$$\Lambda_4 = 20 - \frac{39}{77}k^2 + \frac{52674}{29674645}k^4 - \frac{2805228}{527822\,910615}k^6 + \frac{3776\,48326\,11642}{74\,924536\,936711\,587125}k^8 + \cdots,$$

for $m = 5$:

$$\Lambda_5 = 30 - \frac{59}{117}k^2 + \frac{63338}{56056455}k^4 - \frac{15200884}{13\,045065\,812415}k^6 + \frac{239\,831300\,295254}{4156\,307659\,965616\,462125}k^8 + \cdots,$$

for $m = 6$:

$$\Lambda_6 = 42 - \frac{83}{165}k^2 + \frac{60134}{76366125}k^4 - \frac{1010\,60636}{27651\,60111\,65625}k^6 + \frac{59256\,56959\,98458}{395\,96554\,97527\,57828\,59375}k^8 + \cdots$$

Generally, we have obtained

$$\begin{aligned}\Lambda_m = m(m+1) &- \frac{2m^2+2m-1}{(2m-1)(2m+3)}\,k^2 + \frac{k^4}{2}\left(\frac{(m-1)^2m^2}{(2m-3)(2m-1)^3(2m+1)}\right.\\ &\left.- \frac{(m+1)^2(m+2)^2}{(2m+1)(2m+3)^3(2m+5)}\right)\\ + k^6&\left(\frac{(m+1)^2(m+2)^2}{(2m-1)(2m+1)(2m+3)^5(2m+5)(2m+7)}\right.\\ &\left.- \frac{(m-1)^2m^2}{(2m-5)(2m-3)(2m-1)^5(2m+1)(2m+3)}\right)\\ + k^8&\left[2\left(\frac{(m-1)^2m^2}{(2m-5)^2(2m-3)(2m-1)^7(2m+1)(2m+3)^2}\right.\right.\\ &\left.\left.- \frac{(m+1)^2(m+2)^2}{(2m-1)^2(2m+1)(2m+3)^7(2m+5)(2m+7)^2}\right)\right.\\ &+ \frac{1}{16}\left(\frac{m^2(m-1)^2(m-2)^2(m-3)^2}{(2m-7)(2m-5)^2(2m-3)^3(2m-1)^4(2m+1)}\right.\\ &\left.- \frac{(m+1)^2(m+2)^2(m+3)^2(m+4)^2}{(2m+1)(2m+3)^4(2m+5)^3(2m+7)^2(2m+9)}\right)\\ &+ \frac{1}{8}\left(\frac{(m+1)^4(m+2)^4}{(2m+1)^2(2m+3)^7(2m+5)^2} - \frac{m^4(m-1)^4}{(2m-3)^2(2m-1)^7(2m+1)^2}\right)\\ &\left.+ \frac{1}{2}\left(\frac{(m-1)^2m^2(m+1)^2(m+2)^2}{(2m-3)(2m-1)^4(2m+1)^2(2m+3)^4(2m+5)}\right)\right] + O(k^{10})\end{aligned}$$

For the sake of numerical calculations the following results may be useful. They give six decimals accuracy for $|k^2| \leq 1$. For greater values of $|k|$ (not too large) they may yield adequate approximations λ_m, which can further be made more accurate by the method of the preceding section.

$\Lambda_0 =$	0.000000	000000		$\Lambda_1 =$	2.000000	000000	
	−0.333333	333333	k^2		−0.600000	000000	k^2
	−0.014814	814815	k^4		−0.006857	142857	k^4
	−0.000470	311581	k^6		+0.000060	952381	k^6
	+0.000013	586779	k^8		+0.000002	589628	k^8
	+0.000002	428096	k^{10}		−0.000000	087280	k^{10}
	+0.000000	056788	k^{12}		−0.000000	001558	k^{12}
	+····				+····		

$\Lambda_2 =$	6.000000	000000		$\Lambda_3 =$	12.000000	000000	
	−0.523809	523810	k^2		−0.511111	111111	k^2
	+0.010150	091783	k^4		+0.003294	176331	k^4
	+0.000476	081183	k^6		−0.000059	598927	k^6

$$
\begin{array}{llll}
 & -0.000014 & 108911 & k^8 \\
 & -0.000002 & 437821 & k^{10} \\
 & +\cdots\cdot & & \\
\end{array}
\qquad
\begin{array}{llll}
 & -0.000002 & 654223 & k^8 \\
 & +\cdots\cdot & & \\
 & & & \\
\end{array}
$$

$$
\begin{array}{llll}
\Lambda_4 = & 20.000000 & 000000 & \\
 & -0.506493 & 506494 & k^2 \\
 & +0.001775 & 050721 & k^4 \\
 & -0.000005 & 314714 & k^6 \\
 & +0.000000 & 504038 & k^8 \\
 & +\cdots\cdot & & \\
\end{array}
\qquad
\begin{array}{llll}
\Lambda_5 = & 30.000000 & 000000 & \\
 & -0.504273 & 504274 & k^2 \\
 & +0.001129 & 896637 & k^4 \\
 & -0.000001 & 165259 & k^6 \\
 & +0.000000 & 057703 & k^8 \\
 & +\cdots\cdot & & \\
\end{array}
$$

$$
\begin{array}{llll}
\Lambda_6 = & 42.000000 & 000000 & \\
 & -0.503030 & 303030 & k^2 \\
 & +0.000787 & 443385 & k^4 \\
 & -0.000000 & 365478 & k^6 \\
 & +0.000000 & 014965 & k^8 \\
 & +\cdots\cdot & & \\
\end{array}
$$

Finally, we may notice that, during the preparation of the present paper, we found a paper by Sandeman (23) giving the development of Λ_0 in powers of k^2 up to k^{14}, included. This k^{14} term is

$$-\frac{2894282}{3^{12}\cdot 5^6\cdot 7^3\cdot 11\cdot 13}\,k^{14} = -0.000000\ 007106\ k^{14}$$

Compare also Johnson (13).

5. Tables for spheroidal wave functions of order zero. Fairly extensive tables for spheroidal wave functions are now available (26) for the range $-25 \leq k^2 \leq 25$; e.g. prolate spheroidal functions (k^2 negative), and oblate spheroidal functions (k^2 positive) both for order zero and for orders 1, 2, 3.

In our thesis (6) data were given for oblate spheroidal wave functions of order zero up to $k^2 = 100$. During war time our calculations were largely extended, and checked. We have now stopped our numerical work in view of the existence of the aforementioned American tables, although all the necessary auxiliary functions α_n, β_n, etc., were calculated for the first-order functions too.

The following data, concerning spheroidal wave functions of order zero and first kind, may be considered as supplementary to those of (26).

Table I shows the characteristic values Λ_0, Λ_1, Λ_2, for integral values of k^2, $-10 \leq k^2 \leq 10$.

Table II contains Λ_3, Λ_4, Λ_5, Λ_6 for positive integral values of $k^2 \leq 10$.

Table III gives the coefficients b_n of the normalized characteristic function $X_0(\xi)$ for $-10 \leq k^2 \leq 10$.

The first odd characteristic function $X_1(\xi)$ can be obtained from table IV for the same range of k^2.

Tables V–IX show the coefficients of the functions X_2, X_3, $\cdots$, X_6, for positive integral $k^2 \leq 10$.

TABLE I

k^2	A_0	A_1	A_2
−10	2.305040	7.285254	11.790394
−9	2.136732	6.820888	11.192939
−8	1.959207	6.342739	10.594773
−7	1.771184	5.850492	9.997253
−6	1.571156	5.343904	9.401958
−5	1.357357	4.822809	8.810735
−4	1.127734	4.287129	8.225713
−3	0.879934	3.736870	7.649318
−2	0.611314	3.172128	7.084258
−1	0.319000	2.593085	6.533473
0	0	2	6
1	−0.348602	1.393206	5.486800
2	−0.729392	0.773098	4.996484
3	−1.144328	0.140119	4.531027
4	−1.594493	−0.505244	4.091509
5	−2.079934	−1.162478	3.677958
6	−2.599668	−1.831051	3.289357
7	−3.151841	−2.510421	2.923796
8	−3.733982	−3.200049	2.578732
9	−4.343293	−3.899400	2.251269
10	−4.976896	−4.607952	1.938420

TABLE II

k^2	A_3	A_4	A_5	A_6
0	12	20	30	42
1	11.492121	19.495277	29.496855	41.497757
2	10.990438	18.994079	28.995964	40.997089
3	10.494513	18.496395	28.497321	40.497988
4	10.003864	18.002228	28.000923	40.000458
5	9.517981	17.511597	27.506763	39.504497
6	9.036338	17.024541	27.014846	39.010106
7	8.558395	16.541109	26.525161	38.517282
8	8.083615	16.061383	26.037710	38.026027
9	7.611465	15.585448	25.552488	37.536339
10	7.141428	15.113424	25.069492	37.048221

TABLE III

Characteristic function $X_0(\xi) = \sum_0^\infty b_{2n} P_{2n}(\xi)$

λ^2	b_0	b_2	b_4	b_6	b_8	b_{10}
−10	0.944709	−0.728578	0.110455	−0.007498	0.000288	−0.000007
−9	0.951472	−0.684479	0.094690	−0.005827	0.000203	−0.000005
−8	0.958380	−0.635639	0.079247	−0.004365	0.000136	−0.000003
−7	0.965363	−0.581441	0.064298	−0.003120	0.000085	−0.000001
−6	0.972311	−0.521212	0.050067	−0.002097	0.000049	−0.000001
−5	0.979071	−0.454254	0.036840	−0.001294	0.000025	
−4	0.985428	−0.379882	0.024958	−0.000706	0.000011	
−3	0.991099	−0.297493	0.014835	−0.000316	0.000004	
−2	0.995716	−0.206682	0.006949	−0.000100	0.000001	
−1	0.998846	−0.107374	0.001824	−0.000013		
0	1.000000					
1	0.998691	0.114368	0.001976	0.000014		
2	0.994509	0.233927	0.008138	0.000118	0.000001	
3	0.987210	0.356205	0.018683	0.000408	0.000005	
4	0.976790	0.478301	0.033563	0.000979	0.000016	
5	0.963507	0.597278	0.052483	0.001914	0.000039	
6	0.947848	0.710493	0.074931	0.003279	0.000079	0.000001
7	0.930440	0.815971	0.100266	0.005114	0.000143	0.000003
8	0.911948	0.912502	0.127799	0.007438	0.000238	0.000005
9	0.892980	0.999612	0.156881	0.010250	0.000369	0.000008
10	0.874065	1.077435	0.186946	0.013533	0.000540	0.000014

TABLE IV

Characteristic function $X_1(\xi) = \sum_0^\infty b_{2n+1} P_{2n+1}(\xi)$

λ^2	b_1	b_3	b_5	b_7	b_9	b_{11}
−10	0.964429	−0.402104	0.046184	−0.002528	0.000081	−0.000002
−9	0.970923	−0.364436	0.037696	−0.001858	0.000054	−0.000001
−8	0.976877	−0.325710	0.029954	−0.001312	0.000034	−0.000001
−7	0.982232	−0.286082	0.023016	−0.000882	0.000020	
−6	0.986936	−0.245730	0.016934	−0.000556	0.000011	
−5	0.990948	−0.204851	0.011752	−0.000322	0.000005	
−4	0.994236	−0.163656	0.007499	−0.000164	0.000002	
−3	0.996784	−0.122359	0.004197	−0.000069	0.000001	
−2	0.998586	−0.081179	0.001852	−0.000020		
−1	0.999651	−0.040326	0.000459	−0.000002		
0	1.000000					
1	0.999664	0.039616	0.000447	0.000002		
2	0.998683	0.078362	0.001764	0.000019		
3	0.997105	0.116098	0.003902	0.000063	0.000001	
4	0.994984	0.152711	0.006812	0.000147	0.000002	
5	0.992373	0.188112	0.010436	0.000281	0.000004	
6	0.989330	0.222236	0.014716	0.000473	0.000009	
7	0.985910	0.255039	0.019595	0.000733	0.000016	
8	0.982167	0.286500	0.025011	0.001066	0.000027	
9	0.978150	0.316612	0.030908	0.001477	0.000042	0.000001
10	0.973908	0.345386	0.037230	0.001969	0.000062	0.000001

TABLE V

Characteristic function $X_2(\xi) = \sum_0^\infty b_{2n} P_{2n}(\xi)$

λ^2	b_0	b_2	b_4	b_6	b_8	b_{10}
0		1.000000				
1	−0.022875	0.998525	0.024445	0.000206	0.000001	
2	−0.046799	0.993846	0.048736	0.000821	0.000007	
3	−0.071286	0.985722	0.072766	0.001840	0.000024	0.000001
4	−0.095772	0.974150	0.096431	0.003258	0.000057	0.000001
5	−0.119671	0.959391	0.119654	0.005067	0.000110	0.000002
6	−0.142464	0.941931	0.142398	0.007260	0.000190	0.000003
7	−0.163759	0.922394	0.164677	0.009837	0.000302	0.000006
8	−0.183310	0.901438	0.186545	0.012799	0.000450	0.000010
9	−0.201017	0.879661	0.208098	0.016150	0.000640	0.000016
10	−0.216892	0.857550	0.229438	0.019902	0.000880	0.000024

TABLE VI

Characteristic function $X_3(\xi) = \sum_0^\infty b_{2n+1} P_{2n+1}(\xi)$

λ^2	b_1	b_3	b_5	b_7	b_9	b_{11}
0		1.000000				
1	−0.016979	0.999565	0.017626	0.000118		
2	−0.033587	0.998287	0.035224	0.000470	0.000003	
3	−0.049768	0.996217	0.052770	0.001057	0.000012	
4	−0.065475	0.993406	0.070253	0.001877	0.000027	
5	−0.080671	0.989910	0.087661	0.002929	0.000053	
6	−0.095328	0.985786	0.104979	0.004212	0.000092	0.000001
7	−0.109432	0.981091	0.122202	0.005725	0.000145	0.000002
8	−0.122970	0.975879	0.139316	0.007465	0.000217	0.000004
9	−0.135939	0.970201	0.156318	0.009433	0.000308	0.000006
10	−0.148343	0.964106	0.173200	0.011626	0.000422	0.000010

TABLE VII

Characteristic function $X_4(\xi) = \sum_0^\infty b_{2n} P_{2n}(\xi)$

λ^2	b_0	b_2	b_4	b_6	b_8	b_{10}	b_{12}
0			1.000000				
1	0.000091	−0.013588	0.999768	0.013773	0.000076		
2	0.000368	−0.027140	0.999074	0.027528	0.000304	0.000002	
3	0.000834	−0.040653	0.997918	0.041266	0.000684	0.000006	
4	0.001493	−0.054128	0.996300	0.054977	0.001215	0.000015	
5	0.002348	−0.067563	0.994218	0.068651	0.001897	0.000029	
6	0.003404	−0.080957	0.991669	0.082279	0.002729	0.000051	0.000001
7	0.004663	−0.094312	0.988648	0.095855	0.003712	0.000080	0.000001
8	0.006125	−0.107546	0.984408	0.109286	0.004839	0.000120	0.000002
9	0.007806	−0.120900	0.981162	0.122811	0.006122	0.000171	0.000003
10	0.009695	−0.134130	0.976680	0.136173	0.007550	0.000234	0.000005

TABLE VIII

Characteristic function $X_0(\xi) = \sum_0^\infty b_{2n+1} P_{2n+1}(\xi)$

k^2	b_1	b_3	b_5	b_7	b_9	b_{11}	b_{13}
0			1.000000				
1	0.000068	−0.011218	0.999854	0.011294	0.000046		
2	0.000273	−0.022423	0.999418	0.022584	0.000213	0.000001	
3	0.000611	−0.033609	0.998690	0.033863	0.000478	0.000004	
4	0.001080	−0.044772	0.997673	0.045127	0.000850	0.000009	
5	0.001681	−0.055906	0.996367	0.056373	0.001327	0.000018	
6	0.002408	−0.067006	0.994772	0.067596	0.001910	0.000031	
7	0.003262	−0.078066	0.992888	0.078793	0.002598	0.000050	0.000001
8	0.004236	−0.089080	0.990719	0.089958	0.003391	0.000073	0.000001
9	0.005331	−0.100043	0.988263	0.101088	0.004290	0.000105	0.000002
10	0.006543	−0.110949	0.985522	0.112180	0.005292	0.000143	0.000003

TABLE IX

Characteristic function $X_0(\xi) = \sum_0^\infty b_{2n} P_{2n}(\xi)$

k^2	b_0	b_2	b_4	b_6	b_8	b_{10}	b_{12}	b_{14}
0				1.000000				
1		0.000050	−0.009535	0.999899	0.009571	0.000039		
2	−0.000001	0.000202	−0.019061	0.999597	0.019140	0.000157	0.000001	
3	−0.000004	0.000454	−0.028580	0.999094	0.028702	0.000353	0.000003	
4	−0.000010	0.000805	−0.038087	0.998390	0.038257	0.000627	0.000006	
5	−0.000020	0.001256	−0.047578	0.997486	0.047802	0.000980	0.000012	
6	−0.000035	0.001807	−0.057050	0.996381	0.057331	0.001410	0.000020	
7	−0.000056	0.002458	−0.066501	0.995076	0.066845	0.001919	0.000032	
8	−0.000084	0.003207	−0.075928	0.993572	0.076341	0.002505	0.000048	0.000001
9	−0.000120	0.004055	−0.085326	0.991867	0.085816	0.003169	0.000068	0.000001
10	−0.000165	0.005001	−0.094694	0.989966	0.095268	0.003910	0.000094	0.000002

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NOTE ON C. J. BOUWKAMP'S PAPER "ON SPHEROIDAL WAVE FUNCTIONS OF ORDER ZERO"

By Gertrude Blanch

With the great amount of research which gets published, both here and abroad, an author's conscience is not entirely clear on the duty to read *everything* which any one else has ever published on his subject. In this present instance, however, I seem to be entirely blameless, as the available literature does not mention the work of Bouwkamp at all. By the "available literature" I mean papers which are known well enough to be reviewed in "Mathematical Reviews," for instance. I have scanned the index of this periodical since 1941, and nowhere is the thesis of Dr. Bouwkamp mentioned. No one else in this country or in England who is working on Mathieu functions seems to have known about the very interesting work of Dr. Bouwkamp, and so, while I gladly concede to him priority in discovery, it is still true that my work was done independently and published at a time when no knowledge existed on the work done by Dr.Bouwkamp—that is, no knowledge of his paper in this country and England.

To the extent that my paper deals entirely with Mathieu functions, the results are not duplicated in the work of Dr. Bouwkamp. For although the central idea is the same in my work as in his, the specific relations for Mathieu functions have to be set down with care, as the first two recurrence relations are not the same as the general ones, in the case of Mathieu functions. The same difficulty does not seem to exist in the case of spheroidal wave functions. Moreover, in my paper some simple means are given for obtaining by recurrence the "R's" on which the final solution for the error is based. The summary of Bouwkamp does not give similar aids, although it may very well be that in his more extenisve thesis some similar devices are given. In any event, such computing aids, while very important from the viewpoint of the reader, are of course of secondary importance to the central ideal on which the method is based.

Mathematical Tables Project
National Bureau of Standards

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ON THE BUCKLING OF ELASTIC RINGS

By G. F. Carrier

1. Introduction. In the theory of stability and in the closely related theory of the elastica, there still remain various untreated problems dealing with thin elastic rings. The present paper deals with four such problems which seem of (at least academic) interest. One of these illustrates a somewhat new point of view. Specifically, we consider the elastica problem of the buckled circular ring under uniform radial pressure, the stability problem of closed symmetric rings under any loading which leads to a stability (eigenvalue) problem, the stability problem of a ring which has been pre-stressed into a configuration with large deflections, and the stability of a ring subjected to a uniform pressure applied harmonically in time. Some features of the eigenvalue problems which arise differ in an interesting manner from the more classical forms.

2. The General Equations. The equations of dynamic equilibrium of a curved segment of an elastic strip which is sufficiently thin that the elongation of its center line may be completely neglected[1] are readily expressed by equating the moment about the center of curvature, the moment about the point a (Fig. 1), and the radial force, to the corresponding inertia terms. These equations are

$$R'\rho + t\rho - M' = z\rho A_t \tag{1}$$

$$M' - T = 0 \tag{2}$$

$$-R\varphi' + q - T' = zA_r \tag{3}$$

Here R, q, T, M, t, as defined in Fig. 1, are the various stress resultants and forces acting on the segment; ρ is the radius of curvature of the segment in the deformed shape; A_t, A_r, are the components of acceleration referred to the distorted coordinate system; z is the mass per unit length; and s is the Lagrangian coordinate taken as the distance along the ring. We adopt such units of length that the points $s = 0$ and $s = 2\pi$ coincide for any closed ring under consideration. Hence, for an originally circular ring, s becomes synonomous with the angular coordinate θ. In these equations, primes indicate differentiation with regard to s. Furthermore, q and t are always taken to remain normal and tangent respectively to the central fiber of the ring.

The original shape of the ring may be defined completely, as is well known, by specifying the curvature in the undistorted configuration. We call this initial curvature $1/\rho_0(s)$.

We may eliminate T and R among Eqs. (1), (2), (3) and obtain

$$(q\rho)' + t - (M''\rho)' - M'/\rho = \rho z[(\rho A_r)' + A_t]. \tag{4}$$

This equation rigorously defines the motion of the "inextensible" elastic ring.

[1] The limiting process by which this becomes justified is given in [2].

We note, before proceeding with specific problems, that M may be written as

$$M = EI(1/\rho_0 - 1/\rho) = EI(1/\rho_0 - g) \tag{5}$$

(i.e., here and in the following we set $g = 1/\rho$). Thus for initially circular rings, Eq. (4) becomes

$$[(q/g)' + t]/EI + (g''/g)' + (g^2/2)' = z/EIg[(Ar/g)' + A_t]. \tag{6}$$

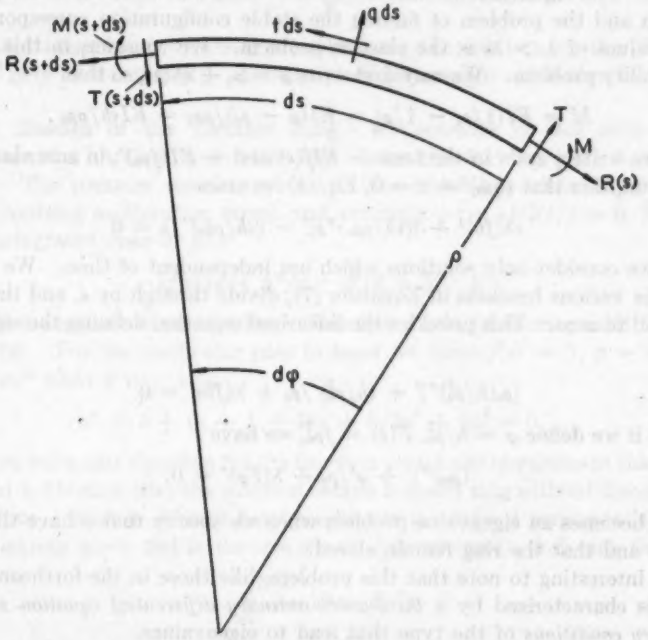

FIG. 1. Forces and stress resultants on element of deformed ring

3. The Generalized Problem of Static Ring Stability. We now consider a closed ring (nor necessarily circular[2]) which is subjected to an externally applied static loading $q(s)$, $t(s)$ [see Fig. (1)], such that $[(q\rho_0)' + t]$ vanishes. It is readily verified that this loading is one such that, by a linear theory, the only deformation is a fiber shortening along the center line which we have agreed to neglect (see, e.g., Eq. 4). Actually, if we considered a more general class of loadings, there would usually exist for any given q, t, a factor λ such that λq, λt, would produce neutral instability. Since, however, the deforma-

[2] Biezeno and Koch have carried out this generalization for special case of the circular ring; see [1].

tions would constitute the solution to a non-linear problem, the determination of λ would be especially difficult. When $[(q\rho_0)' + t]$ vanishes however, we see that for small q and t, the stable solution is always the trivial one $\rho = \rho_0$, $M = 0$. *Hence the determination of λ_0 such that the loading $\lambda_0 q$, $\lambda_0 t$ will produce instability requires only the solution of the linearized eigenvalue problem* (i.e., the problem where we consider only small changes in the original curvature λ_0). In other words, we must ask what multiple λ_0 of q and t leads to a stress system such that there exist equilibrium configurations adjacent to that configuration defined by $\rho = \rho_0$. The eigenvalue problem (i.e., the determination of λ_0) is the stability problem and the problem of finding the stable configuration corresponding to given values of $\lambda > \lambda_0$ is the elastica problem. We consider, in this section, the stability problem. We may first write $\rho = \rho_0 + \epsilon h(s)$, so that

$$M = EI(1/\rho_0 - 1/\rho) = EI(\rho - \rho_0)/\rho\rho_0 = EI\epsilon h/\rho\rho_0 .$$

When we write q and t in the form $-EIf(s)$ and $+EI(f\rho_0)'$, in accordance with the assumption that $(q\rho_0)' + t = 0$, Eq. (4) becomes

$$\epsilon\lambda(fh)' + [(\epsilon h/\rho\rho_0)''\rho]' - (\epsilon h/\rho\rho_0)'/\rho = 0 \tag{7}$$

where we consider only solutions which are independent of time. We now expand the various brackets in Equation (7), divide through by ϵ, and then allow ϵ to tend to zero. This provides the linearized equation defining the eigenvalue λ_0

$$[\rho_0(h/\rho_0^2)'']' + (h/\rho_0^2)'/\rho_0 + \lambda(fh)' = 0.$$

Finally if we define $\varphi = h/\rho_0^2$, $F(s) = f\rho_0^2$, we have

$$(\rho_0\varphi'')' + \varphi'/\rho_0 + \lambda(F\varphi)' = 0. \tag{8}$$

This becomes an eigenvalue problem when we specify that φ have the period 2π in s and that the ring remain closed.

It is interesting to note that this problem, like those in the forthcoming sections, is characterized by a *third-order ordinary differential equation* and *three boundary conditions* of the type that lead to eigenvalues.

The foregoing eigenvalue problem can be put in a form which is frequently more convenient for the actual determination of λ_0. We assume for this derivation that the ring and the loading are doubly symmetric.[3] In Equation (8), we replace $\lambda(F\varphi)'$ by the function

$$G(s) = \mathfrak{S}_1(s - s_0) + \mathfrak{S}_1(s + s_0) + \mathfrak{S}_1(s + \pi - s_0) + \mathfrak{S}_1(s + \pi + s_0),$$

where $\mathfrak{S}_1$ is the conventional singularity function. We note that this corresponds physically to replacing our distributed load by four self-equilibrating concentrated loads. We shall, of course, use the Green's function which arises as a solution to the modified Eq. (8) in order to define a solution to our original problem. This Green's function which can be found either by a Fourier series

[3] i.e., $\rho_0(s) = \rho_0(-s) = \rho_0(\pi - s)$.

expansion or numerically will be called $K(s,s_0)$. Now consider the original problem. For a loading (i.e., the third term in Eq. (8)) given by $\lambda[F\varphi]'$, the solution φ is given by

$$\varphi(s) = \int_0^{\pi/2} \lambda F(s_0)\varphi(s_0)K(s, s_0)\, ds_0$$

or

$$u(s) = \sqrt{F(s)}\,\varphi = \lambda \int_0^{\pi/2} u(s_0)K_1(s, s_0)\, ds_0. \tag{9}$$

A rapidly convergent iteration procedure for the solution of this integral equation[4] may be found in [1]. An example is also carried out in that reference.

4. The Elastica of the Circular Ring. We consider in this section the deformation of a buckled inextensible thin elastic circular ring under a uniform pressure. The pressure remains normal to the center-line when the ring deforms. Omitting acceleration terms, and writing $g = -\lambda f/EI$, $t = 0$, Eq. (6) may be integrated once to give

$$-\lambda f(s)/g + g''/g + g^2/2 = \tfrac{1}{2} - \beta \tag{10}$$

where β is a constant of integration. This formula is, of course, valid for any loading $f(s)$. For the particular case in hand we write $f(s) = 1$, $\beta = \lambda - \delta$ and obtain,* when $g = 1 + u$,

$$u'' + \delta + (\lambda + 1 + \delta)u + 3/2u^2 + \tfrac{1}{2}u^3 = 0. \tag{11}$$

We must solve this equation for the function u with the requirement that for a given load λ, δ is such that the solution defines a closed ring without discontinuities in slope. That is, u must have period π/n in s ($n = 1$ corresponds to the lowest buckling mode and is the one we shall investigate), and such that

$$\int_0^{\pi/n} u\, ds = 0.$$

The linearized problem leads, as is well known, to

$$\delta = 0, \qquad \lambda = 3, \qquad u = A \sin(2s + B),$$

where A, B, are indeterminate.† For $\lambda > 3$, we must retain the term in u^2, u^3, and integrate Eq. (11) as it stands. It is readily shown that the solution of Eq. (11) may be written in the form

[4] In Eq. (9) K_1 indicates $[F(s_0)F(s)]^{\frac{1}{2}}K(s,s_0)$. Note that in the first form of Eq. (9) $(F\varphi)'$ rather than $(F\varphi)'$ appears. This is seen to be valid by noting that when $(F\varphi)$ is a "delta function" $(F\varphi)'$ is an "$\mathfrak{S}_1$" function and $\varphi = K$ which confirms our definition of $K(s\,s_0)$.

* These substitutions for β and g will be useful later.

† This is the lowest mode, i.e. the smallest λ_0. i.e. for $\lambda < 3$ $u = 0$. For $\lambda > 3$ the linearized solution makes no physical sense since u^2/u is not small.

$$u = a[p + q\, cn(\alpha s,k)]/[1 + m\, cn(\alpha s,k)] \tag{12}$$

where cn denotes the elliptic cosine.

If we write $\delta = a\epsilon$, one may obtain relations between the unknown quantities a, p, q, m, ϵ, α, k, as follows. Substitution of the solution Eq. (12) into Eq. (11) gives four relations; the requirement $u(0) = a$, which makes for convenience later is admissable; periodicity π is guaranteed by taking $\alpha = (4/\pi)K(k)$, where K is the complete elliptic integral of the first kind; and $\int_0^{\pi/2} u\, ds = 0$, is the seventh relation. These lead to five non-linear algebraic and two transcendental equations in the foregoing parameters. The method of solution may be found in the appendix. Our results are described in Figure 2 by plots of the change in curvature v.s. the load λ. One may actually obtain the slope, α

$$\alpha = \int_{s_1}^{s} (1 + u)\, ds \tag{13}$$

in closed form (a combination of elliptic integrals of the first and third kinds with other transcendental expressions) but a numerical integration is simpler to perform if one wishes to draw the elastic curve. In any event, it is well known that the foregoing function for the curvature together with the values of $p, q, \cdots$, as given in Table 1, completely describes the configuration.

It should be noted that this problem was treated by M. Lévy [3] but that his solution, also obtained under the assumption of non-extensibility, leads to an inconvenient pair of elliptic integrals with no specified method for finding the quantities analogous to a and δ for a given λ. A numerical procedure applied to his case would be more tedious than was that used here.

5. The Stability of a Pre-Deformed Ring. We shall consider here an originally circular ring which is placed under such a loading that the deformations are large.[5] We wish to introduce the point of view that a measure of the stability of a given configuration of a stressed system may be considered in the following light. We determine a loading of such a nature that it produces no additional *deformation* in the given pre-stressed system,[6] and ask what multiple μ of this loading will produce instability. The larger μ, the more stable the system, i.e., the more additional loading is required to produce instability.

As a particular example, let us consider the ring loaded by $q(s)$ such that $f(s)$ in Eq. (10) is written as

$$\lambda_0 f_0(s) = \lambda_0[m + cn(\alpha s,k) + \epsilon\, cn^2(\alpha s,k)].$$

If we now require that $\beta_0 + 4 = \lambda_0^2(1 - 2k^2)$, $m = (B_0/\lambda_0)$, $\epsilon = \lambda_0/6$, $\alpha^2k^2 = \lambda_0^2/36$, $\alpha = (4/\pi)K(k)$, we have a one parameter family of loadings, and the solution of Equation (6) takes the simple form

$$u_0 = -(\lambda_0/3)\, cn(\alpha s,k)$$

[5] i.e., the problem becomes non-linear.

[6] Without the inextensibility or a comparable assumption one must modify this statement to one restricting the change in deformation to pure fiber shortening, or some similar behavior.

TABLE I

Values of parameters appearing in ring elastica problem

k =	0	.1	.2	.3	.4	.5
p	0	−.0280	−.0540	−.0830	−.115	−.160
q	1.0	.971	.939	.900	.854	.773
m	0	−.0577	−.115	−.184	−.261	−.387
e	0	−.336	−.653	−1.014	−1.38	−1.99
a	0	.475	.959	1.53	2.13	2.98
λ	3.0	3.060	3.227	3.593	4.076	5.408

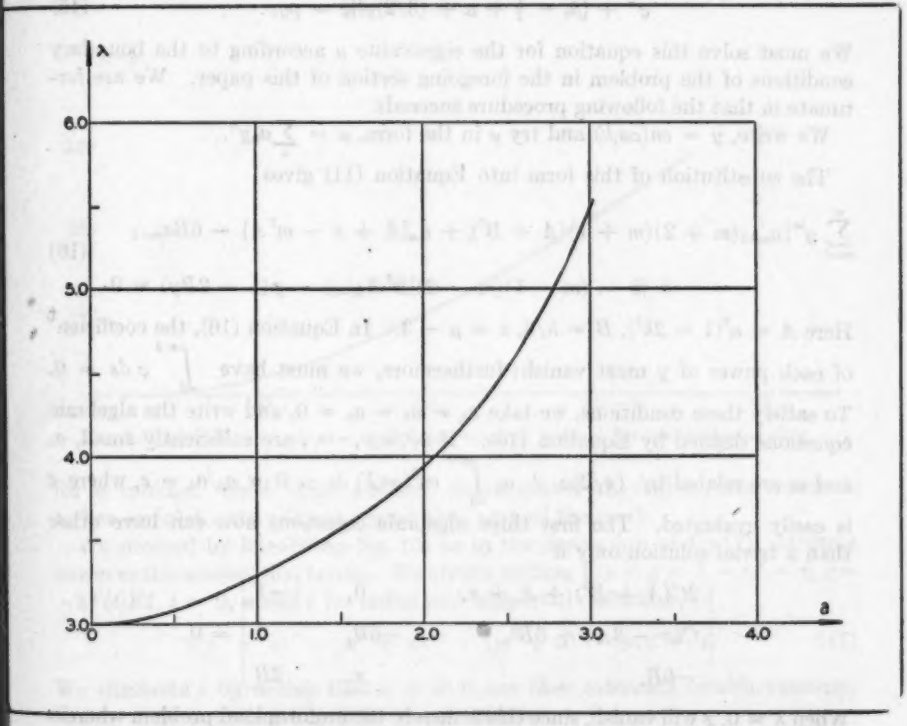

FIG. 2. Eigen loading v.s. change in curvature (see Eq. 12) for ring-elastica problem.

which obeys the necessary boundary conditions. The subscript zero will distinguish this initial configuration from that arising in the buckling problem. Let us now consider a different $f(s)$ as given by

$$\lambda f(s) = \lambda_0 f_0(s) + \mu g_0(s)$$

where

$$g_0 = 1 + u_0(s).$$

Equation (10) becomes, when $g = g_0 + g_1$, $\beta = \beta_0 - \delta$,

$$\lambda_0 f_0 + \mu(g_0 + g_1) + g_0'' + g_1'' + (g_0 + g_1)^3/2 = (\tfrac{1}{2} - \beta_0 + \delta)(g_0 + g_1).$$

In view of the manner by which g_0 was obtained we have

$$g_1'' + [\mu + 3/2g_0^2 - \tfrac{1}{2} + \beta_0]g_1 + 3/2g_0g_1^2 + g_1^3/2 = \delta g_0 + \delta g_1 . \tag{14}$$

Since we are only looking for the eigenvalue μ, the equation may be linearized by writing $g_1 = \epsilon\varphi$, $\delta = \epsilon p$, substituting in Equation (14), dividing by ϵ, and then allowing ϵ to tend to zero. We obtain

$$\varphi'' + [\beta_0 - \tfrac{1}{2} + \mu + (3/2)g_0^2]\varphi = pg_0 . \tag{15}$$

We must solve this equation for the eigenvalue μ according to the boundary conditions of the problem in the foregoing section of this paper. We are fortunate in that the following procedure succeeds.

We write, $y = cn(\alpha s,k)$ and try φ in the form, $\varphi = \sum_n a_n y^n$.

The substitution of this form into Equation (11) gives

$$\sum_{m=0}^{\infty} y^m\{a_{m+2}(m+2)(m+1)(A+B^2) + a_m[A + x - m^2A] - 6Ba_{m-1} + [6 - (m-1)(m-2)]B^2A_{m-2}\} - p(1 - 2By) \equiv 0. \tag{16}$$

Here $A = \alpha^2(1 - 2k^2)$, $B = \lambda/6$, $x = \mu - 3$. In Equation (16), the coefficient of each power of y must vanish; furthermore, we must have $\int_0^{\pi/2} \varphi\, ds = 0$. To satisfy these conditions, we take $a_3 = a_4 = a_5 = 0$, and write the algebraic equations defined by Equation (16). If a_3, a_{10}, $\cdots$, are sufficiently small, a_0 and a_2 are related by, $(\pi/2)a_0 + a_2 \int_0^{\pi/2} cn^2(\alpha s,k)\, ds \simeq 0$ or $a_2/a_0 = c$, where c is easily evaluated. The first three algebraic equations now can have other than a trivial solution only if

$$\begin{vmatrix} 2C(A+B^2) + A + x, & 0, & -1 \\ C(x - 3A) + 6B^2, & -6B, & 0 \\ -6B, & x, & 2B \end{vmatrix} = 0.$$

When $\lambda = 0$, x will vanish, since this is merely the uniform load problem wherein $\mu = 3$. Hence one chooses the smallest root of the foregoing equation. When one solves for a_3/a_0 it is found to be sufficiently small so that no correction need be applied to c at least for λ as large as 3.5.

In Figure 3, the plot of μ v.s. λ indicates that the deformed ring requires a greater average load to cause buckling than does the undeformed ring. Furthermore we find that the configurations obtained by the loadings

$$f(s) = \lambda f_0(s) + \mu(\lambda) g_0(s),$$

are neutrally unstable.

An alternative method (which would correspond to the classical treatment) for testing the stability of this configuration would be to treat Equation (15) with μ and p equal to zero as a problem to determine the eigenvalue λ. Since β_0 and g_0^2 are non-linear in λ, this would be a non-linear eigenvalue problem of considerably greater complexity than the foregoing.

6. The Dynamic Stability of the Circular Ring. We consider in this section the stability of the circular ring under uniform radial pressure whose intensity varies periodically with time. The solution of this problem can be reduced to the form given by Stoker and Lubkin [4] to the analogous problem

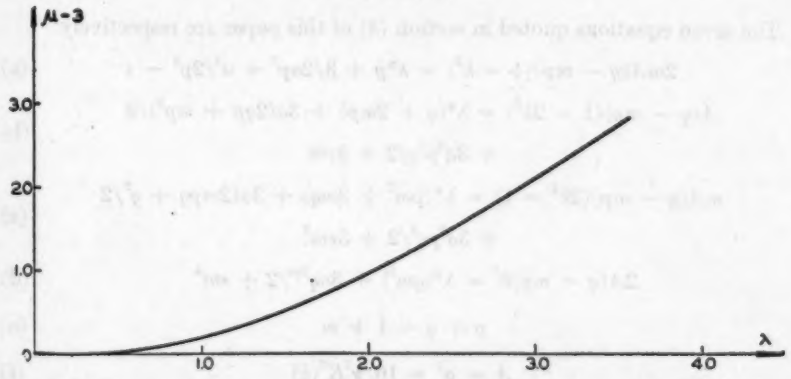

FIG. 3. Eigenvalue vs. parameter defining original loading for pre-deformed ring

for a column. Since these authors' discussion of the subsequent details is rather complete, we will not repeat that part of the work.

We proceed by linearizing Eq. (6) as in the derivation of Eq.(7), retaining however the acceleration terms. We obtain writing $1/\rho = g = 1 - u'' - u$, $q = -\lambda F(t)EI$, $t = 0$, u and v for radial and tangential deflection,

$$\lambda(u'' + u)' - (u'' + u)''' - (u'' + u)' = \mu(\ddot{u}' + \ddot{v}). \tag{17}$$

We eliminate v by noting that $u + v'$ is the fiber extension (which vanishes) and by differentiating Eq. (17) once to obtain

$$\partial^2/\partial s^2[F(t)\lambda(\partial^2/\partial s^2 + 1) + (\partial^2/\partial s^2 + 1)^2]u = \mu(\partial^2/\partial s^2 - 1)\ddot{u}.$$

The instability corresponding to the lowest mode under a static loading is that given by

$$u = A(t) \cos 2s,$$

from which we obtain

$$5\mu\ddot{A} + [12\lambda F(t) - 36]A = 0.$$

If we now let $F(t)$ be harmonic, say

$$F = b + a\cos wt = b + a\cos\theta$$

we may by obvious change of parameter titles obtain

$$\partial^2 A/\partial\theta^2 + (\alpha + \beta\cos\theta)A = 0. \tag{18}$$

This result, as was mentioned previously, has been found in the analogous problem of the column and a complete discussion of the necessary treatment of Equation (18) may be found in [3].

Appendix

The seven equations quoted in section (4) of this paper are respectively

$$2mA(q - mp)(1 - k^2) = \lambda^* p + 3/2ap^2 + a^2/2p^3 - \epsilon \tag{a}$$

$$A(q - mp)(1 - 2k^2) = \lambda^*(q + 2mp) + 3a(2qp + mp^2)/2 + 3a^2p^2q/2 + 3\epsilon m \tag{b}$$

$$mA(q - mp)(2k^2 - 1) = \lambda^*(pm^2 + 2mq) + 3a(2mpq + q^2/2 + 3a^2pq^2/2 + 3\epsilon m^2 \tag{c}$$

$$2A(q - mp)k^2 = \lambda^*(qm^2) + 3aq^2m/2 + \epsilon m^3 \tag{d}$$

$$p + q = 1 + m \tag{e}$$

$$A = \alpha^2 = 16/\pi^2 K^2(k) \tag{f}$$

$$p(1 + m^2/2 + k^2/4 + 3m^4/8 + 9k^4/64 + m^2k^2/16) - q(m/2 + 3m^3/8 + mk^2/16) = 0. \tag{g}$$

Here, the first six equations are exact and we have retained enough terms in Equation (g) for excellent accuracy for the range of admissible values of k. Here $\lambda^* = (\lambda + 1 + \epsilon a)$.

We differentiate these equations with regard to k. It is convenient to regard this as the independent parameter rather than λ since α is explicit in k. Omitting Equation (f) then, we have six quasi-linear equations in the first derivatives of the six unknown quantities. We may obtain the corresponding equations for the second derivatives.

We know that at $k = 0$; $\alpha = 2$, $\lambda^* = 4$, $q = 1$, $a = p = m = \epsilon = 0$. Thus we readily obtain $q'(0)$, $q''(0)$, $p'(0)$, $\cdots$, where primes denote derivatives with regard to k. We then obtain (say) $q(.1) = 1 + .1q'(0) + .005q''(0)$, and the corresponding quantities $p(.1)$, $\cdots$, and may now proceed to use our system of differential equations to find $p'(.1)$, $q'(.1)$, $\cdots$. Then $p(.2) = p(0) + .2p'(.1)$

and so on. When the values (Table 1) obtained are substituted in the original exact equations, the check is excellent.

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BROWN UNIVERSITY

(RECEIVED FEBRUARY 11, 1947)

NEW PROOFS OF THE THEOREMS OF KASNER CONCERNING THE INFINITESIMAL CONTACT TRANSFORMATIONS OF MECHANICS*

By John De Cicco

1. *The theorems.* We submit new proofs of the theorems of Kasner concerning the infinitesimal contact transformations of general dynamics.[1] The theorems deal with the nature of two dynamical systems of the same number of degrees of freedom, for which the commutator or alternant of the associated infinitesimal contact transformations is a point transformation. The main result states that this situation can arise when and only when the expressions for the kinetic energy are the same or differ by a factor which depends only on the n generalized coordinates. The final result is the following converse proposition. Two infinitesimal contact transformations with the same transversality law will have a point transformation for commutator if and only if they are associated with two dynamical systems of the type described above.

We shall prove these theorems for homogeneous contact transformations. The methods are those of the tensor calculus.

2. *Infinitesimal contact transformations.* In this section, we shall collect the various concepts and formulas about infinitesimal contact transformations which are necessary to prove the theorems of Kasner.

An element $(x;p) = (x^i;p_i) = (x^1, \cdots, x^n; p_1, \cdots, p_n)$ of a space V_n of n dimensions consists of a point $(x) = (x^i) = (x^1, \cdots, x^n)$ and a covariant vector $(p) = (p_i) = (p_1, \cdots, p_n)$. A set of ∞^{n-1} elements forms a *union* if the condition $p_i dx^i = 0$, is satisfied. Thus in the case of a union, if the points (x^i) describe the $(n-1)$ dimensional manifold $F(x) = F(x^1, \cdots, x^n) = 0$, then the p_i being proportional to the partial derivatives $\partial F/\partial x^i$, are the components of the covariant normal to the manifold.

A transformation of elements

$$X^i = X^i(x;p), \qquad P_i = P_i(x;p), \tag{1}$$

where the jacobian-matrix is of rank $2n$ in a certain region of $2n$ dimensional space $(x;p)$, is called a *homogeneous contact transformation* if and only if the differential expression $p_i dx^i$, is invariant.[2] The necessary and sufficient conditions for a homogeneous contact transformation are

$$P_i \frac{\partial X^i}{\partial x^j} = p_j, \qquad P_i \frac{\partial X^i}{\partial p_j} = 0. \tag{2}$$

These are all the transformations of elements converting unions into unions.

A one-parameter group G_1 of element transformations is generated by the infinitesimal transformation

$$\delta x^i = X^i - x^i = \xi^i(x;p)\delta t, \qquad \delta p_i = P_i - p_i = \eta_i(x;p)\delta t, \tag{3}$$

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where δt is an infinitesimal. The necessary and sufficient conditions that this G_1 be a one-parameter group of homogeneous contact transformations is that the infinitesimal transformation (3) be of the contact type. Therefore upon substituting (3) into (2), it is found that (3) is an infinitesimal homogeneous contact transformation if and only if

$$p_i \frac{\partial \xi^i}{\partial x^j} + \eta_j = 0, \qquad p_i \frac{\partial \xi^i}{\partial p_j} = 0 \tag{4}$$

Upon setting $p_i \xi^i = \Gamma(x;p)$, we find from these equations that Γ is homogeneous in the p's of degree one and that the ξ^i and η_i are given by the expressions

$$\xi^i = \frac{\partial \Gamma}{\partial p_i}, \qquad \eta_i = -\frac{\partial \Gamma}{\partial x^i}. \tag{5}$$

Moreover, along the trajectories of the group G_1 ; Γ is the same constant. For since the trajectories of (3) are defined by $dx^i = \xi^i\, dt$, $dp_i = n_i\, dt$, it follows by (5) that $d\Gamma = -\eta_i dx^i + \xi^i\, dp_i = 0$. Hence the p's may be so chosen that Γ is unity along all the trajectories of the group G_1 except for the singular trajectories.

An infinitesimal homogeneous contact transformation is defined by the equations

$$\delta x^i = \frac{\partial \Gamma}{\partial p_i}\,\delta t, \qquad \delta p_i = -\frac{\partial \Gamma}{\partial x^i}\,\delta t, \tag{6}$$

where the characteristic function $\Gamma = \Gamma(x;p)$ is homogeneous of degree unity, and the p's are so chosen that Γ is one along the non-singular trajectories of the one-parameter group G_1 generated by this infinitesimal transformation. Moreover the symbol UF of this group G_1 is

$$UF = \frac{\partial \Gamma}{\partial p_\alpha}\frac{\partial F}{\partial x^\alpha} - \frac{\partial \Gamma}{\partial x^\alpha}\frac{\partial F}{\partial p_\alpha} = (\Gamma, F). \tag{7}$$

Now suppose we have two different one-parameter groups of contact transformations with characteristic functions Γ_1 and Γ_2 and with symbols $U_1F = (\Gamma_1, F)$ and $U_2F = (\Gamma_2, F)$ respectively. The formation of the *commutator or alternant* $(U_1, U_2)F = (U_1U_2)F - (U_2U_1)F$ yields the following result.

The commutator $(U_1, U_2)F = (U_1U_2)F - (U_2U_1)F$, of the symbols $U_1F = (\Gamma_1, F)$ and $U_2F = (\Gamma_2, F)$ representing two one-parameter groups of homogeneous contact transformations with characteristic functions $\Gamma_1(x;p)$ and $\Gamma_2(x;p)$ respectively, is the symbol of another one-parameter group of homogeneous contact transformations. The characteristic function $\Gamma_3(x;p)$ of this new group is

$$\Gamma_3 = (\Gamma_1, \Gamma_2) = \frac{\partial \Gamma_1}{\partial p_\alpha}\frac{\partial \Gamma_2}{\partial x^\alpha} - \frac{\partial \Gamma_1}{\partial x^\alpha}\frac{\partial \Gamma_2}{\partial p_\alpha}. \tag{8}$$

Of course, Γ_3 is homogeneous of degree one in the p's.

For, by (7) it is seen that the commutator is

$$(U_1, U_2)F = \left(\frac{\partial\Gamma_1}{\partial p_b}\frac{\partial^2\Gamma_2}{\partial p_a \partial x^b} + \frac{\partial\Gamma_2}{\partial x^b}\frac{\partial^2\Gamma_1}{\partial p_a \partial p_b} - \frac{\partial\Gamma_1}{\partial x^b}\frac{\partial^2\Gamma_2}{\partial p_a \partial p_b} - \frac{\partial\Gamma_2}{\partial p_b}\frac{\partial^2\Gamma_1}{\partial p_a \partial x^b}\right)\frac{\partial F}{\partial x^a}$$

$$(9)\qquad - \left(\frac{\partial\Gamma_1}{\partial p_b}\frac{\partial^2\Gamma_2}{\partial x^a \partial x^b} + \frac{\partial\Gamma_2}{\partial x^b}\frac{\partial^2\Gamma_1}{\partial x^a \partial p_b} - \frac{\partial\Gamma_1}{\partial x^b}\frac{\partial^2\Gamma_2}{\partial x^a \partial p_b} - \frac{\partial\Gamma_2}{\partial p_b}\frac{\partial^2\Gamma_1}{\partial x^a \partial x^b}\right)\frac{\partial F}{\partial p_a}.$$

Thus the coefficient of $\partial F/\partial x^a$ is the partial derivative of $\Gamma_3 = (\Gamma_1, \Gamma_2)$ of (8) with respect to p_a and the coefficient of $\partial F/\partial p_a$ is the negative of the partial derivative of $\Gamma_3 = (\Gamma_1, \Gamma_2)$ of (8) with respect to x^a. This establishes the fact that $\Gamma_3 = (\Gamma_1, \Gamma_2)$ is the characteristic function of the new infinitesimal transformation.

3. *The infinitesimal contact transformation associated with a dynamical system.* This subject was developed by Lie[3] and later studied by Vessiot[4]. In this section, we shall give a survey of some of the results needed for our work.

Consider a conservative dynamical system with n degrees of freedom. Its potential energy V is a function of the n generalized coordinates x^i, that is, $V = V(x)$; and its kinetic energy T is a quadratic form in the n generalized velocity components, that is, T is of the form

$$T = \tfrac{1}{2} g_{ij}\dot{x}^i\dot{x}^j, \tag{10}$$

where $(\dot{x}^1, \cdots, \dot{x}^n)$ are the components of the contravariant velocity vector and the $g_{ij}(x)$ are the components of a symmetric covariant tensor of second order for which $|g_{ij}| \neq 0$. The sum of the potential energy V and the kinetic energy T is constant, that is,

$$T + V = E, \tag{11}$$

where E is the constant of energy.

Let g^{ij} be the quotient of the cofactor of g_{ij} in the matrix (g_{ij}) by the determinant $|g_{ij}|$, so that $g^{ia}g_{aj} = \delta^i_j$. If $g_{ij}\dot{x}^j = p_i$, then $g^{ij}p_ip_j = g_{ij}\dot{x}^i\dot{x}^j$. Finally let $\lambda(x) = [2(E - V)]^{-1/2}$ so that λ is a function of (x).

The function $\Gamma(x;p)$ defined by the equation

$$\Gamma = \lambda(x)[g^{ij}p_ip_j]^{1/2}, \tag{12}$$

is the characteristic function of the infinitesimal contact transformation

$$\delta x^i = \lambda^2 g^{ij}p_j\,\delta s,\quad \delta p_i = -\tfrac{1}{2}\frac{\partial}{\partial x^i}(\lambda^2 g^{jk})p_jp_k\,\delta s, \tag{13}$$

where the parameter s is defined by

$$ds^2 = \frac{1}{\lambda^2}g_{ij}\,dx^i\,dx^j. \tag{14}$$

The transformation (13) is called *the infinitesimal contact transformation associated with the dynamical system.* The form (14) is preferred to the usual form: $ds^2 = g_{ij}\,dx^i\,dx^j$ as it allows for the interpretation into dynamical types.

The trajectories of the dynamical system corresponding to a fixed value of the energy E form a *natural family* and are in one-to-one correspondence with the geodesics of the riemannian manifold with the fundamental form (14).

As defined by Kasner, the direction determined by the vector of components (δx^i) connecting the point (x) of any element $(x;p)$ and the point (X) of the transformed element $(X;P)$ under any infinitesimal contact transformation (6) is said to be *transversal*[5] to the direction (p). For the transformation (13), this transversality condition is orthogonality with respect to the riemannian manifold (14). This condition of orthogonality is equivalent to stating that each point is carried into an infinitesimal geodesic hypersphere. These hyperspheres are of equal radius if $\lambda = 1$ (or constant), in which case (13) is called a *generalized infinitesimal dilatation*. Finally it is noted that the transversality of (13) is involutorial.

4. *Proof of the main theorem of Kasner concerning the infinitesimal contact transformations of mechanics.* We shall find it convenient to discuss this main result as two separate theorems, firstly the direct part, and secondly the converse.

Let $\Gamma(x;p)$ be homogeneous of degree one in the p's and let $\lambda(x)$ and $\mu(x)$ be arbitrary functions of (x). These functions λ and μ are not related to the λ of (12) of the preceding section. The functions

$$\Gamma_1 = \lambda(x)\Gamma(x;p), \qquad \Gamma_2 = \mu(x)\Gamma(x;p), \tag{15}$$

are the characteristic functions of two infinitesimal contact transformations with the symbols $U_1F = (\lambda\Gamma_1, F)$ and $U_2F = (\mu\Gamma, F)$ respectively. By (8), the commutator $(U_1, U_2)F$ represents an infinitesimal contact transformation with characteristic function

$$\Gamma_3 = (\Gamma_1, \Gamma_2) = \left(\lambda \frac{\partial \mu}{\partial x^s} - \mu \frac{\partial \lambda}{\partial x^s}\right) \Gamma \frac{\partial \Gamma}{\partial p_s}. \tag{16}$$

Theorem 1. If two dynamical systems lead to the same expression for the kinetic energy, the commutator of the associated infinitesimal contact transformations is a point transformation.

For by hypothesis, it is deduced from (12) that the characteristic functions of the associated infinitesimal contact transformations are given by (15) where $\Gamma^2 = g^{ij}p_ip_j$. Hence substituting this value of Γ into (16), we find that the characteristic function Γ_3 of the commutator is

$$\Gamma_3 = (\Gamma_1, \Gamma_2) = \left(\lambda \frac{\partial \mu}{\partial x^s} - \mu \frac{\partial \lambda}{\partial x^s}\right) (g^{si} p_i). \tag{17}$$

Since this is linear in the p's, it follows by (6) that the commutator represents a point transformation.

Theorem 2. If the commutator of the associated infinitesimal contact transformations of two dynamical systems is a point transformation, then the expressions for the kinetic energy of the two dynamical systems are the same or differ merely by a factor which depends only on the n generalized coordinates.

From (12), it is seen that the characteristic functions of the associated contact transformations of two dynamical systems may be assumed in the forms

$$\Gamma_1^2 = g^{ij}p_ip_j, \qquad \Gamma_2^2 = \bar{g}^{ij}p_ip_j. \tag{18}$$

By (8), the characteristic function Γ_3 of the commutator is

$$\Gamma_3 = \frac{1}{2\Gamma_1\Gamma_2}\left(g^{ia}\frac{\partial \bar{g}^{jk}}{\partial x^a} - \bar{g}^{ia}\frac{\partial g^{jk}}{\partial x^a}\right)p_ip_jp_k. \tag{19}$$

For this to be the characteristic function of a point transformation, Γ_3 must be linear integral in the p's. Hence it is necessary that Γ_3 be rational in the p's. Assuming that neither Γ_1 nor Γ_2 represents a point transformation (which cases have no dynamical interest), it is seen from (19) that the quadratic forms in (18) can differ only by a factor, that is, $\bar{g}^{ij}(x) = \lambda(x)g^{ij}(x)$. Therefore the two dynamical systems are such that the expressions for the kinetic energy are the same or differ merely by a factor.

Theorems 1 and 2 together constitute the main theorem of Kasner which was first stated and proved in 1910. Kasner showed that Theorem 1 may be restated in geometrical language as follows.

Theorem 3. If two infinitesimal contact transformations lead to the same linear involutorial law of transversals, the commutator is a point transformation.

This follows from Theorem 1 and the fact that the transversality defined by $\delta x^i = \lambda^2 g^{ij}p_j\delta s$, is unchanged when λ is changed in value.

5. *Pairs of infinitesimal contact transformations with the same transversality laws and with point transformations for commutators.* We shall state and prove the following proposition of Kasner.

Theorem 4. Two infinitesimal contact transformations with the same transversality law will have a point transformation for commutator if and only if they are associated with two dynamical systems for which the expressions for the kinetic energy are the same or differ merely by a factor.

First of all we shall prove that if two infinitesimal contact transformations with the characteristic functions $\Gamma(x;p)$ and $\bar{\Gamma}(x;p)$ have the same transversality law, then $\bar{\Gamma} = \lambda(x)\Gamma$. For by (6), we must have

$$\frac{\partial \bar{\Gamma}}{\partial p_a}\Big/\frac{\partial \Gamma}{\partial p_a} = \frac{\partial \bar{\Gamma}}{\partial p_b}\Big/\frac{\partial \Gamma}{\partial p_b} \quad \text{for} \quad a, b = 1, \cdots, n. \tag{20}$$

Hence by the theory of jacobians, we have $\bar{\Gamma} = F(x;\Gamma)$. Since Γ and $\bar{\Gamma}$ are homogeneous of degree one in the p's, it follows by Euler's theorem on homogeneous functions that

$$p_a\frac{\partial \bar{\Gamma}}{\partial p_a} = p_a\frac{\partial \Gamma}{\partial p_a}F_\Gamma = \Gamma F_\Gamma = F. \tag{21}$$

Hence $F = \lambda(x)\Gamma$ so that $\bar{\Gamma} = \lambda(x)\Gamma$.

The characteristic function Γ_3 of the commutator of the infinitesimal contact transformations whose characteristic functions are $\Gamma(x;p)$ and $\bar{\Gamma} = \lambda(x)\Gamma(x;p)$, is

$$\Gamma_3 = \tfrac{1}{2}\frac{\partial\lambda}{\partial x^a}\frac{\partial}{\partial p_a}\Gamma^2. \tag{22}$$

Therefore the commutator will represent a point transformation if and only if

$$\frac{\partial\lambda}{\partial x^a}\frac{\partial^3\Gamma^2}{\partial p_a\partial p_b\partial p_c} = 0. \tag{23}$$

Suppose at least one of the third order partial derivatives $\partial^3\Gamma^2/\partial p_a\partial p_b\partial p_c$, is not zero. Then (23) defines a linear homogeneous partial differential equation of first order in the function λ. By standard theory, this has $(2n - 1)$ functionally independent solutions, and any other solution is a function of these. Now evidently $(p_1, \cdots, p_n)$ are n functionally independent solutions. Hence there exist $(n - 1)$ functionally independent solutions $\lambda_1(x), \cdots, \lambda_{n-1}(x)$, depending on (x) only. Replacing λ by each of these values, it follows that $\partial^3\Gamma^2/\partial p_a\partial p_b\partial p_c$ are proportional to functions of (x) only. But $\partial^2\Gamma^2/\partial p_b\partial p_c$ is homogeneous of degree zero so that by Euler's theorem on homogeneous functions, we have

$$p_a\frac{\partial^3\Gamma^2}{\partial p_a\partial p_b\partial p_c} = 0. \tag{24}$$

Hence since the third order partial derivatives of Γ with respect to the p's are proportional to functions of (x) only, it follows that they must vanish identically. This is a contradiction of our hypothesis.

By the preceding argument, it follows that

$$\frac{\partial^3\Gamma^2}{\partial p_a\partial p_b\partial p_c} = 0 \quad \text{for} \quad a, b, c = 1, 2, \cdots, n. \tag{25}$$

Hence Γ^2 is a homogeneous quadratic form in the p's. From this, Theorem 4 follows.

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A HEURISTIC EXPOSITION OF WIENER'S MATHEMATICAL THEORY OF PREDICTION AND FILTERING

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Consider the function of time $f(t)$ which is the sum of a function $g(t)$ and a disturbance, $f(t) - g(t)$. How can we best extract $g(t)$ from $f(t)$? This is the problem of filtering.

More generally how can we best determine $g(t + h)$ from $f(t - \tau), 0 \leqq \tau < \infty$? If $h > 0$ we have here a problem in prediction as well as in filtering.

In case $f(t) = g(t)$ then there is no problem of filtering but there may still be a prediction problem, i.e. finding of $f(t + h)$ from $f(t - \tau), 0 \leqq \tau < \infty$.*

An explicit solution to this problem was given by N. Wiener in 1942 in a document not publicly available. Here we shall present an expository account of Wiener's linear theory making several minor departures from Wiener's procedure. Moreover we shall deal mainly with the analytic rather than the statistical aspects.

The theory developed here will apply if the function, $f(t)$, possesses an auto-correlation function,

$$\phi(t) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t + \tau) f(\tau)\, d\tau;$$

if $g(t)$ possesses an auto-correlation function $\gamma(t)$; and if the cross correlation function, $\chi(t) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} g(t + \tau) f(\tau)\, d\tau$ exists. The importance of the auto-correlation function will be seen from Sec. 1 which follows. In case $f(t) = g(t)$ then $\chi(t) = \gamma(t) = \phi(t)$. We shall further assume that $\phi(t)$ and $\chi(t)$ are continuous and that each has a Fourier transform. These requirements eliminate from the scope of the theory such functions as $f(\tau) = 1, -\infty < \tau \leq t$ or $f(\tau) = \sin a\tau, -\infty < \tau \leq t$. Each of these functions has a $\phi(t)$ which does not tend to zero as $|t| \to \infty$ and therefore has no Fourier transform. In fact, these requirements exclude all elementary functions. However, all the elementary functions are perfectly predictable and therefore their exclusion involves no real loss.

It is necessary to subtract the perfectly predictable component from a function, $f(t)$, before applying the theory presented here. Thus if the average of $f(t)$ is not zero it should be subtracted from $f(t)$.

1. The Autocorrelation Function. In the linear theory of prediction and filtering we attempt to express $g(t + h)$ in terms of a linear combination of values of $f(t - \tau)$ where $\tau \geq 0$. One way of doing this would be to select several values of τ, $\tau_n \geq 0$, and try to choose coefficients, a_n, so that

* For further background on this problem see the introduction in an earlier paper of the author, *The Wiener RMS Error Criterion in Filter Design and Prediction*, J. Math. and Phys. *25*, 261–278, 1946.

$$\sum_0^N a_n f(t - \tau_n) \tag{1.0}$$

gives an optimum prediction of $g(t + h)$. This procedure has much to recommend it in practice and is easy to carry through. A more general procedure is to attempt to predict the value of $f(t + h)$ by means of

$$\int_0^\infty f(t - \tau)\, dK(\tau). \tag{1.1}$$

This latter expression involves only the past of $f(t)$ since $\tau \geq 0$ in (1.1).

Another example of an operation on the part of $f(t)$ is

$$\sum_0^N a_n f^{(n)}(t - \tau_n), \qquad \tau_n \geq 0, \tag{1.2}$$

where $f^{(n)}$ denotes the nth derivative of f.

It will be convenient for purposes of exposition to use the form

$$\int_0^\infty f(t - \tau)K(\tau)\, d\tau \tag{1.3}$$

which appears to be more restrictive than (1.1) and not to include (1.2). Actually the method of treatment used will be such that the result will come out as an operator on $f(t - \tau)$. This operator may be of the form of (1.3) but it also can be more general in nature and can include the cases (1.1) and (1.2). Thus our assumed form (1.3) is no real restriction.

The question we ask is: How shall we choose the operator $K(\tau)$ so that for a prescribed $h \geq 0$,

$$g(t + h) - \int_0^\infty f(t - \tau)K(\tau)\, d\tau \tag{1.4}$$

is as small as possible? Before answering this question we must decide what we mean by the phrase "as small as possible." Here we shall mean that the average (with respect to t) of the square of (1.4) should be a minimum. To be precise, let

$$I[K] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left[g(t + h) - \int_0^\infty f(t - \tau)K(\tau)\, d\tau \right]^2 dt. \tag{1.5}$$

Our question now becomes: What choice of $K(\tau)$ will make I a minimum?

If we expand the right member of (1.5) we get, inverting limits freely,

$$\begin{aligned} I[K] = {} & \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g^2(t + h)\, dt \\ & -2 \int_0^\infty K(\tau)\, d\tau \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(t + h)f(t - \tau)\, dt \\ & + \int_0^\infty K(\tau_1)\, d\tau_1 \int_0^\infty K(\tau_2)\, d\tau_2 \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t - \tau_1)f(t - \tau_2)\, dt, \end{aligned} \tag{1.6}$$

the last term of which arises from the

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_0^\infty K(\tau)f(t-\tau)\,d\tau\right)^2 dt. \tag{1.7}$$

Incidentally, since (1.7) is non-negative, it follows that the last term of (1.6) is also non-negative.

The right-hand member of (1.6) becomes considerably simplified if we introduce the autocorrelation functions ϕ, χ and γ. A consequence of the existence of $\phi(t)$ is that for any a, b, and c

$$\phi(t-c)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T+a}^{T+b} f(t+\tau)f(\tau+c)\,d\tau.$$

A similar result is true for χ and γ. This last result can be used to show that $\phi(t)$ and $\gamma(t)$ are even functions. We also use it to rewrite (1.6) as

$$\begin{aligned} I[K] &= \gamma(0) - 2\int_0^\infty K(\tau)\chi(h+\tau)\,d\tau \\ &\quad + \int_0^\infty K(\tau_1)\,d\tau_1\int_0^\infty K(\tau_2)\phi(\tau_1-\tau_2)\,d\tau_2 . \end{aligned} \tag{1.8}$$

Since our problem is to find $K(\tau)$ so as to minimize I, we see from (1.8) that the question of what K to choose so as to get an optimum value for $g(t+h)$ *does not depend on $f(t)$ and $g(t)$ directly but rather on $\phi(t)$ and $\chi(t)$*, the correlation functions. This is a most important point. Finding K depends on knowing two statistical functions of f and g rather than on knowing f and g themselves. If we find two ensembles of functions $\{f(t)\}$ and $\{g(t)\}$, having the same correlation functions, ϕ, γ and χ, then we can choose a $K(\tau)$ that will give us the best prediction of $g(t+h)$ in terms of the past of $f(t)$ for every g and f in the respective ensembles.

Since the last term in (1.6) is non-negative it follows that the last term in (1.8) to which it is equal must also be non-negative. Thus for any K and any autocorrelation function,

$$\int_0^\infty K(\tau_1)\,d\tau_1\int_0^\infty K(\tau_2)\phi(\tau_1-\tau_2)\,d\tau \geq 0. \tag{1.9}$$

2. The Integral Equation. If $K(\tau)$ actually makes I a minimum this means that replacing $K(\tau)$ by $K(\tau)+\epsilon M(\tau)$, where ϵ is a real number and $M(\tau)$ is a function of τ, must increase I. That is

$$I(K+\epsilon M)\geqq I(K).$$

From (1.8) we have

$$\begin{aligned} I[K+\epsilon M] = I[K] &- 2\epsilon\int_0^\infty \chi(\tau+h)M(\tau)\,d\tau \\ &+ 2\epsilon\int_0^\infty M(\tau)\,d\tau\int_0^\infty K(\tau_1)\phi(\tau-\tau_1)\,d\tau_1 \\ &+ \epsilon^2\int_0^\infty M(\tau)\,d\tau\int_0^\infty M(\tau_1)\phi(\tau-\tau_1)\,d\tau_1 . \end{aligned}$$

Or

$$I[K + \epsilon M] = I[K] - 2\epsilon J_1 + \epsilon^2 J_2 \tag{2.0}$$

where

$$J_1 = \int_0^\infty M(\tau)\left(\chi(\tau + h) - \int_0^\infty \phi(\tau - \tau_1)K(\tau_1)\,d\tau_1\right)d\tau$$

and

$$J_2 = \int_0^\infty M(\tau)\,d\tau \int_0^\infty \phi(\tau - \tau_1)M(\tau_1)\,d\tau_1 .$$

Now if for some $M(\tau)$, $J_1 \neq 0$, then, by changing the sign of $M(\tau)$ if necessary, we have $J_1 > 0$. Writing (2.0) as

$$I[K + \epsilon M] = I[K] - 2\epsilon(J_1 - \tfrac{1}{2}\epsilon J_2), \tag{2.1}$$

we see that since $J_1 > 0$ we can, by making ϵ small enough, make $J_1 - \frac{1}{2}\epsilon J_2 > 0$. Thus (2.1) gives us

$$I[K + \epsilon M] < I[K]$$

which is impossible. Therefore $J_1 = 0$ for any $M(\tau)$.

Clearly we will have $J_1 = 0$ for any $M(\tau)$ if

$$\chi(t + h) - \int_0^\infty \phi(t - \tau)K(\tau)\,d\tau = 0, \qquad t \geqq 0. \tag{2.2}$$

It is important to note that (2.2) need hold only for $t \geqq 0$ since $M(\tau) \equiv 0$, $\tau < 0$. Conversely, the fact that $J_1 = 0$ for any $M(\tau)$ implies (2.2). Thus if $K(\tau)$ minimizes $I[K]$, then (2.2) must hold.

With (2.2) valid, $J_1 = 0$ and (2.0) becomes

$$I[K + \epsilon M] = I[K] + \epsilon^2 J_2 .$$

As we saw in (1.9), $J_2 \geqq 0$. This implies that $I[K + \epsilon M] \geqq I[K]$. We see then that (2.2) is not only a necessary but is also a sufficient condition for $I[K]$ to be a minimum. The problem has thus been reduced to the solution of the integral equation (2.2) for the function K.*

* In the integral equation (2.2), K is unknown and χ and ϕ are known. The equation might be called a Wiener-Hopf integral equation of the first kind. In the Wiener-Hopf equation itself it is necessary to restrict the kernel ϕ to be exponentially small in magnitude. When transforms are taken this provides a strip in the complex plane in which to match up factors. In the present case no such strip is available and, as will be seen, factorization is carried out on the real axis of the complex plane. The W-H equation of the first kind also arises in some problems of electromagnetic theory as has been shown by Schwinger. [See for example, the Reflection of an Electromagnetic Plane Wave by an Infinite Set of Plates I, by J. F. Carlson and A. E. Heins, Quarterly of Appl. Math. 4, p. 313, (Jan. 1947).] However in the equation discussed by Carlson and Heins, there is available a strip of regularity for matching factors. Therefore their method of solution follows that of the original W-H equation.

3. The Modified Integral Equation. Since the second term in (2.2) is in the form of a convolution, it is natural to conclude that this equation can be solved for $K(\tau)$ by use of the Fourier transform theorem. However, because of the requirement that (2.2) holds only for $t \geqq 0$, this conclusion is false. To see precisely why the Fourier transform does not work, let us try it.

Multiplying both sides of (2.2) by e^{iut} and integrating for $t > 0$ we have

$$\int_0^\infty \chi(t+h)e^{iut}\,dt = \int_0^\infty e^{iut}\,dt \int_0^\infty K(\tau)\phi(t-\tau)\,d\tau. \tag{3.0}$$

But the right member above is equal to

$$\int_0^\infty K(\tau)\,d\tau \int_0^\infty e^{iut}\phi(t-\tau)\,dt.$$

Setting $t = s + \tau$, we get for (3.0)

$$\int_0^\infty \phi(t+h)e^{iut}\,dt = \int_0^\infty K(\tau)e^{iu\tau}\,d\tau \int_\tau^\infty e^{ius}\phi(s)\,ds. \tag{3.1}$$

Now in the usual case the limits in the last integral would not involve τ but would be fixed. In that case the last equation could be solved for $\int_0^\infty K(\tau)e^{iu\tau}\,d\tau = k(u)$ from which $K(\tau)$ can be determined. Here however there is no simplification. Nevertheless, since $\tau \geqq 0$, *notice that the last integral would not involve τ if* $\phi(t) = 0$, $t < 0$. Of course this last requirement is impossible, but the general idea can be exploited as we shall now proceed to do.

We replace ϕ by a function which vanishes for negative t. This is achieved as follows. We introduce the functions $\psi_1(t)$ and $\psi_2(t)$ such that

$$\psi_1(t) = 0, \quad t < 0, \tag{3.2}$$

$$\psi_2(t) = 0, \quad t > 0, \tag{3.3}$$

$$\phi(t) = \int_{-\infty}^\infty \psi_1(\tau)\psi_1(t-\tau)\,d\tau. \tag{3.4}$$

Of course it is necessary to show that this is possible. This we shall do later. Using (3.4) in (2.2) and also using (3.3) we get

$$\chi(t+h) = \int_{-\infty}^0 \psi_2(\tau)\,d\tau \int_0^\infty \psi_1(t-\tau-s)K(s)\,ds, \qquad t > 0. \tag{3.5}$$

Now if it is possible to find an $\alpha(t)$ such that

$$\chi(t) = \int_{-\infty}^0 \alpha(t-\tau)\psi_2(\tau)\,d\tau, \tag{3.6}$$

then (3.5) becomes

$$\int_{-\infty}^0 \alpha(t+h-\tau)\psi_2(\tau)\,d\tau = \int_{-\infty}^0 \psi_2(\tau) \int_0^\infty \psi_1(t-\tau-s)K(s)\,ds, \qquad t > 0.$$

From this we find

$$\int_{-\infty}^0 \psi_2(\tau)\left[\alpha(t+h-\tau) - \int_0^\infty \psi_1(t-\tau-s)K(s)\,ds\right]d\tau = 0, \qquad t \geqq 0.$$

Clearly this equation, and therefore also (2.2), will hold if

$$\alpha(t + h - \tau) - \int_0^\infty \psi_1(t - \tau - s)K(s)\,ds = 0, \qquad t > 0, \qquad \tau < 0.$$

Or since $t - \tau > 0$, the above is equivalent to

$$\alpha(t + h) = \int_0^\infty \psi_1(t - s)K(s)\,ds, \qquad t > 0. \tag{3.7}$$

Therefore, we have only to solve (3.7) for $K(\tau)$ in order to minimize I. The equation (3.7) has the same form as (2.2) except that $\psi_1(t) = 0$, $t < 0$, and consequently (3.7) *will yield to the Fourier transform method.*

Of course everything depends on our being able to find a ψ_1, and ψ_2 satisfying (3.2), (3.3) and (3.4). We observe that (3.4) is an integral equation of the ordinary convolution type which we want to solve for two functions ψ_1 and ψ_2 subject to auxiliary conditions (3.2) and (3.3). Since (3.4) and (3.6) involve ordinary convolutions, they can be simplified by use of the Fourier transform.

4. The Factorization Problem. We proceed now to find ψ_1, ψ_2 and α. Once this is done, solving (3.7) for K will be a simple and routine Fourier transform problem. Let

$$\int_{-\infty}^\infty \phi(t)e^{iut}\,dt = \Phi(u). \tag{4.0}$$

Then by the Fourier transform theorem

$$\phi(t) = \frac{1}{2\pi}\int_{-\infty}^\infty \Phi(u)e^{-iut}\,du. \tag{4.1}$$

Similarly if

$$\int_0^\infty \psi_1(t)e^{iut}\,dt = \Psi_1(u), \tag{4.2}$$

then

$$\psi_1(t) = \frac{1}{2\pi}\int_{-\infty}^\infty \Psi_1(u)e^{-iut}\,du, \tag{4.3}$$

and an analogous result holds for $\psi_2(t)$. Multiplying (3.4) by e^{iut} and integrating we have

$$\Phi(u) = \int_{-\infty}^\infty \psi_2(\tau)\,d\tau \int_{-\infty}^\infty e^{iut}\,\psi_1(t - \tau)\,dt.$$

Setting $t - \tau = s$ we get

$$\Phi(u) = \Psi_1(u)\,\Psi_2(u). \tag{4.4}$$

The equation (4.2) gives us, if $w = u + iv$,

$$\Psi_1(w) = \int_0^\infty \psi_1(t)e^{-vt}e^{iut}\,dt, \tag{4.5}$$

and

$$\Psi_1'(w) = \int_0^\infty it\psi_1(t)e^{-vt}e^{iut}\,dt. \tag{4.6}$$

In the upper half-plane $v > 0$, $\Psi_1'(w)$ is determined as a finite function, since for $v > 0$, the term e^{-vt} in (4.6) assures the convergence of the integral. Thus $\Psi_1(w)$ defined by (4.5) is an analytic function of w in the upper half-plane $v > 0$. Also $\Psi_1(w)$ is a bounded function in the upper half-plane $v \geqq 0$. We observe then that *the Fourier transform of a function vanishing over* $(-\infty, 0)$ *is analytic and bounded in the upper half-plane.* Also the Fourier transform of a function vanishing over $(0, \infty)$ is analytic and bounded in the lower half-plane.

The converse of this result is also true. For suppose that $\Psi_1(w)$ is analytic and bounded. Then the integral

$$\psi_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi_1(w)e^{-iwt}\,dw$$

can be shown to be zero for $t < 0$ simply by closing the path of integration in the upper half-plane and using Cauchy's integral theorem. The fact that e^{vt} is small for $t < 0$ and large v makes this step legitimate. Thus we conclude that if a *function is analytic and bounded in the upper half-plane its Fourier transform vanishes over* $(-\infty, 0)$. Similarly the Fourier transform of a function analytic and bounded in the lower half-plane vanishes over $(0, \infty)$.

Combining this fact with (4.4), $\Phi(u) = \Psi_1(u)\Psi_2(u)$, we see that the *problem of finding* $\psi_1(t)$ *is reduced to the problem of factoring* $\Phi(u)$ *into two factors,* $\Psi_1(u)$ *and* $\Psi_2(u)$, *such that* $\Psi_1(u + iv)$ *is analytic and bounded in the upper half-plane* $v > 0$, *and* $\Psi_2(u + iv)$ *is analytic and bounded in the lower half-plane,* $v < 0$.

Before attempting to factor $\Phi(u)$, we observe that $\Phi(u) \geqq 0$. [This result is established in Wiener's theory of Generalized Harmonic Analysis, Acta Math. 55, pp. 117–258, 1930. In fact $\Phi(u)$ is the density of the energy of $f(t)$ at frequency u. Thus it must be positive.] If $\Psi_1(u + iv) = P(u, v) + iQ(u, v)$ is analytic for $v > 0$, then it follows at once from the Cauchy-Riemann equations, that $\Psi_2(u + iv) = P(u, -v) - iQ(u, -v)$ is analytic for $v < 0$. Moreover using the bar to denote the conjugate complex number we see that $\Psi_1(u) = \bar{\Psi}_2(u)$, so that $\Psi_1(u)\Psi_2(u) \geq 0$. In other words, by choosing $\Psi_2(u, v)$ as $P(u, -v) - iQ(u, -v)$ we satisfy the requirement $\Phi(u) > 0$. Moreover since

$$\Psi_1(u)\bar{\Psi}_2(u) = |\Psi_1(u)|^2 \equiv \Phi(u)$$

we see that

$$|\Psi_1(u)| = \sqrt{\Phi(u)}. \tag{4.7}$$

Thus the problem of finding $\psi_1(t)$ *has now been reduced to the problem of finding* $\Psi_1(u + iv)$ *analytic and bounded in the upper half-plane* $v \geqq 0$ *knowing the value of* $|\Psi_1(u)|$.

5. The Functions ψ_1, ψ_2, and α. We introduce the function

$$\lambda(w) = \log \Psi_1(w).$$

If we write $\lambda(w)$ in terms of its real and imaginary parts, then $\lambda(u + iv) = p(u, v) + iq(u, v)$. The requirements on Ψ_1 are certainly fulfilled if $\lambda(w)$ is analytic for $v > 0$, if $e^{p(u, v)}$ is bounded for $v > 0$, and if

$$p(u, 0) = \tfrac{1}{2} \log \Phi(u). \tag{5.0}$$

This last requirement is (4.7). The condition that $\lambda(w)$ be analytic for $v > 0$, is equivalent to the condition that $p(u, v)$ be a harmonic function for $v > 0$. In this way all our requirements can be specified in terms of $p(u, v)$.

The determination of the harmonic function, $p(u, v)$, taking on specified values on the real axis, as is indicated in (5.0), is well known. In fact

$$p(u, v) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\frac{1}{2} v \log \Phi(s)}{(u - s)^2 + v^2}\, ds \tag{5.1}$$

will be harmonic* and will satisfy (5.0). The integral (5.1) is the well known Poisson Integral, and $e^{p(u, v)}$ is a bounded function for $v > 0$. Thus all requirements on $p(u, v)$ are fulfilled. We shall also have occasion to use the fact that, with $p(u, v)$ determined by (5.1), $e^{-p(u, v)}$ is very limited in magnitude for $v > 0$. (It is certainly $0(e^{\epsilon |w|})$ for any $\epsilon > 0$.)

If R denotes "real part of," then

$$\frac{v}{(u - s)^2 + v^2} = R \frac{i}{w - s}.$$

Thus (5.1) can be written as

$$p(u, v) = R \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{\frac{1}{2} \log \Phi(s)}{w - s}\, ds.$$

Taking account of the fact that $\Phi(s)$ is an even function we have

$$p(u, v) = R \frac{i}{\pi} \int_{0}^{\infty} \frac{w \log \Phi(s)}{w^2 - s^2}\, ds. \tag{5.2}$$

Since $p(u, v) = R\lambda(w)$ it follows from (5.2) that

$$\lambda(w) = \frac{i}{\pi} \int_{0}^{\infty} \frac{w \log \Phi(s)}{w^2 - s^2}\, ds. \tag{5.3}$$

We recall that

$$\Psi_1(w) = e^{\lambda(w)}.$$

Thus we have completely determined Ψ_1, and with it, from (4.3), also $\psi_1(t)$. Not only is $\Psi_1(w)$ analytic and bounded in the upper half-plane but $1/\Psi_1(w) = e^{-\lambda(w)}$ is also analytic in the upper half-plane. This last relationship, which appears to be incidental here, is in fact a basic requirement as we shall see.

* If the integral, (5.1), diverges then Wiener has shown $f(t)$ can be predicted perfectly from its past.

We turn next to (3.6)

$$\chi(t) = \int_{-\infty}^{0} \psi_2(\tau)\alpha(t-\tau)\,d\tau. \tag{5.4}$$

Introducing the Fourier transforms

$$\mathrm{A}(u) = \int_{-\infty}^{0} \alpha(t)\,e^{iut}\,dt \text{ and } \mathrm{X}(u) = \int_{-\infty}^{\infty} \chi(t)\,e^{iut}\,dt$$

we get from (5.4) $\mathrm{X}(u) = \mathrm{A}(u)\Psi_2(u)$. Thus $\mathrm{A}(u) = \mathrm{X}(u)/\Psi_2(u)$ is determined. We find $\alpha(t)$ from $\alpha(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{A}(u)\,e^{-iut}\,du$. Thus $\psi_1(t)$, $\psi_2(t)$ and $\alpha(t)$ are determined.

6. The Prediction Operator. We return now to the equation (3.7),

$$\alpha(t+h) = \int_0^\infty \psi_1(t-\tau)K(\tau)\,d\tau, \qquad t \geqq 0.$$

Multiplying the equation by e^{iwt} and integrating for $t \geqq 0$ we find

$$\int_0^\infty \alpha(t+h)e^{iwt}\,dt = \int_0^\infty K(\tau)\,d\tau \int_0^\infty \psi_1(t-\tau)e^{iwt}\,dt.$$

Or setting $t - \tau = s$,

$$\int_0^\infty \alpha(t+h)e^{iwt}\,dt = \int_0^\infty K(\tau)e^{iw\tau}\,d\tau \int_0^\infty \psi_1(s)e^{iws}\,ds. \tag{6.0}$$

If for $v \geqq 0$,

$$k(w) = \int_0^\infty K(\tau)e^{iw\tau}\,dt \tag{6.1}$$

where $w = u + iv$, then we have from (6.0)

$$k(w) = \int_0^\infty \alpha(t+h)e^{iwt}\,dt/\Psi_1(w). \tag{6.2}$$

We have thus found the Fourier transform of $K(\tau)$.

In order that $K(\tau)$ as determined from its Fourier transform, $k(w)$, shall be null for $t < 0$, it is necessary for $k(w)$ to be analytic and of limited growth in the upper half w-plane. Now the integral on the right side of (6.2) defines a bounded analytic function for $v \geqq 0$. As has already been indicated by the remark in the paragraph following (5.1), $e^{-\lambda(w)}$ is limited in magnitude in the upper half-plane. Thus $k(w)$ as given by (6.2) is analytic and limited in magnitude in the upper half-plane [$0(e^{\epsilon|w|})$ for any $\epsilon > 0$]. We may conclude, therefore, that $K(\tau)$ as determined from $k(w)$ will be null for $\tau < 0$. Strictly speaking $k(u)$ often will not have a Fourier transform in the orthodox sense but is rather the transform of an operator which deals only with the interval $(0, \infty)$.

For example the best representation of $g(t+h)$ may be given by $f'(t)$ and

not by $\int_0^\infty f(t-\tau)K(\tau)\,d\tau$ at all. In this case if we apply $\left[-\frac{d}{d\tau}\right]_{\tau=0}$ to $f(t-\tau)$ we get $f'(t)$ as desired. If we apply $\left[-\frac{d}{d\tau}\right]_{\tau=0}$ to $e^{iw\tau}$ we get $-iw$. Thus in this case we would find $k(w) = -iw$.

Again if $f(t-a)$ should be used in the representation of $g(t)$ this is the result of taking $[f(t-\tau)]_{\tau=a}$.
Doing this to $e^{iw\tau}$ we get e^{iwa}. More generally then if we find as a result of using (6.2) that

$$k(w) = -w^2 + 2iwe^{iw} - e^{2iw} + \frac{i}{w+i},$$

then in place of an operation on $f(t-\tau)$ of just the form $\int K(\tau)\,d\tau$ we get

$$\frac{d^2}{d\tau^2}f(t-\tau)\Big]_{\tau=0} + 2\frac{d}{d\tau}f(t-\tau)\Big]_{\tau=1} - f(t-\tau)\Big]_{\tau=1} + \int_0^\infty f(t-\tau)e^{-\tau}\,d\tau$$
$$= f''(t) - 2f'(t-1) - f(t-\tfrac{1}{2}) + \int_0^\infty f(t-\tau)e^{-\tau}\,d\tau.$$

We can check this by using $e^{iw\tau}$ in place of $f(t-\tau)$ getting

$$-w^2 + 2iwe^{iw} - e^{2iw} + \int_0^\infty e^{iw\tau}e^{-\tau}\,d\tau$$

which is in fact $k(w)$.

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AN ALGORITHM FOR LEAST SQUARES

By K. L. Nielsen and L. Goldstein

1. Introduction. The problem of curve fitting by the method of least squares is a classical problem in mathematical statistics. The literature contains numerous methods of solution each designed to reduce the amount of arithmetic. The modern trend as developed by Fisher,[1] Jordan,[2] Aitken,[3] Allen,[4] and others (for a more complete bibliography see (1)) has been along the lines of orthogonal polynomials; wherein the condition of orthogonality yields methods for computing the coefficients of the desired polynomial. The elimination of a great deal of arithmetic has been made possible by the compilation of tables of orthogonal polynomials[5] and the establishment of recurrence formulas. Thus the Fisher method together with his tables seems to be the most efficient method for obtaining a polynomial which best fits a given set of data. However, the method of orthogonal polynomials leaves the impression of an unnecessary step if it is desired to have the polynomial as a "power series" in x, the given argument. For example, the Fisher method is to solve first for the regression coefficients β_j in the polynomial

$$y = \sum_{j=0}^{n} \beta_j \xi_j \tag{1.1}$$

where ξ_j is the set of orthogonal polynomials

$$\xi_j = \sum_{i=0}^{j} a_{ij} x^i. \tag{1.2}$$

and then to solve for a_{ij} by a recurrence formula or by substitution of known values into the function (1.2).

For some problems it may not be necessary to obtain y as a function of x; it may suffice to have y as a function of ξ. If such be the case the Fisher method is probably the most rapid. However, in the case where it is required to have y as a function of x, it may be advantageous to have an alternate method.

It is the purpose of this paper to develop an alternate method for the "complete" solution of this classical problem. A method which does not depend upon orthogonal polynomials but which follows directly from the condition that the

[1] Fisher, R. A., "Statistical Methods For Research Workers", 7th Ed., Oliver and Boyd, London, 1938, Chapter V.

[2] Jordan, Charles, "Approximation and Graduation According to the Principle of Least Squares by Orthogonal Polynomials", Annals of Math. Statistics, vol. 3, (1932), pp. 257–357.

[3] Aitken, A. C., "On the Graduation of Data by the Orthogonal Polynomials of Least Squares", Proc. Royal Soc. of Edinburg, vol. 53, (1932–33) pp. 54–78.

[4] Allen, F. E., "The General Form of the Orthogonal Polynomials for Simple Series With Proofs of Their Simple Properties", Proc. Royal Soc. of Edinburg, vol. 50, (1930), pp. 310–320.

[5] Fisher, R. A. and Yates, F., "Statistical Tables for Biological, Agricultural and Medical Research", Rev. Ed., Oliver & Boyd, London, 1943.

residuals be a minimum. The method yields a numerical computation for the solution of a given problem which is as rapid and as easily preformed as the Fisher method. In fact the operational steps of these two methods are similar in nature and are of the same degree of arithmetic complexity. As in the case of the Fisher method it is applicable only to equally spaced values of the argument.

The central device employed is Crout's algorithm[6] for the solution of a system of linear equations. It will be shown that the application of Crout's algorithm to the "normal" equations of least squares after the data has been centralized about the mean and the interval length has been removed yields a "derived determinant" which is a function of the number of items only. Thus the elements of this determinant may be tabulated; (see Table I). The remaining computations which involve the actual values of the data are few in number and are performed from simple closed formulas.

A summary of the necessary computational steps is given in §6. For greater clarification a few illustrative examples are exhibited in §7.

2. Statement of the Problem. Let there be given a set of m points (x_j, y_j), $(j = 1, \cdots, m)$, and let it be desired to fit this set of points by a polynomial of degree n,

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = \sum_{i=0}^{n} a_i x^i. \tag{2.1}$$

As is well known, the problem is to determine the coefficients a_i, $(i = 0, \cdots, n)$ in such a way that the polynomial (2.1) gives the best fit to the set of data (x_j, y_j), $(j = 1, \cdots, m)$. The standard procedure by the method of least squares is to form the m residual equations and obtain a function in the unknowns a_i, $(i = 0, \cdots, n)$ by summing the squares of these residuals. The condition that this function be a minimum then gives $n + 1$ linear equations in the $n + 1$ unknowns a_i obtained by equating the respective partial derivatives to zero. The problem thus is one of computing from the given data the coefficients of the unknowns a_i in this system of linear equations and then solving this system for the a_i.

3. Transformations. It is now assumed that we have equidistant data so that

$$x_j = x_1 + (j - 1)h, \qquad (j = 1, \cdots, m), \tag{3.1}$$

where h is the interval length between the equidistant x-points. We make the following transformation on the data

$$X_j = x_j - \bar{x}, \qquad Y_j = y_j - \bar{y} \tag{3.2}$$

[6] Crout, P. D., "A Short Method For Evaluating Determinants and Solving Systems of Linear Equations With Real and Complex Coefficients", Transactions of the A. I. E. E., vol. 60, (1941), pp. 1235–41.

TABLE I

m	S_2	A_{13}	A_{33}	A_{24}	A_{44}	A_{15}	A_{35}	A_{25}	A_{55}
5	5.00	2.000000	1.400000	3.40	1.440000	6.800000	4.428571	6.200000	.822857
6	8.75	2.916667	2.133333	5.05	3.702857	14.729167	6.785714	14.476190	4.702041
7	14.00	4.000000	3.000000	7.00	7.714286	28.000000	9.571429	28.714286	16.163265
8	21.00	5.250000	4.000000	9.25	14.142857	48.562500	12.785714	51.142857	43.102041
9	30.00	6.666667	5.133333	11.80	23.760000	78.666667	16.428571	84.333333	98.057143
10	41.25	8.250000	6.400000	14.65	37.440000	120.862500	20.500000	131.200000	199.680000
11	55.00	10.000000	7.800000	17.80	56.160000	178.000000	25.000000	195.000000	374.400000
12	71.50	11.916667	9.333333	21.25	81.000000	253.229167	29.928571	279.333333	658.285714
13	91.00	14.000000	11.000000	25.00	113.142857	350.000000	35.285714	388.142857	1099.102041
14	113.75	16.250000	12.800000	29.05	153.874286	472.062500	41.071428	525.714286	1758.563265
15	140.00	18.666667	14.733333	33.40	204.582857	623.466667	47.285714	696.676190	2714.782041
16	170.00	21.250000	16.800000	38.05	266.760000	808.562500	53.928571	906.000000	4064.914285
17	204.00	24.000000	19.000000	43.00	342.000000	1032.000000	61.000000	1159.000000	5928.000000
18	242.25	26.916667	21.333333	48.25	432.000000	1298.729167	68.500000	1461.333333	8448.000000
19	285.00	30.000000	23.800000	53.80	538.560000	1614.000000	76.428571	1819.000000	11797.028571
20	332.50	33.250000	26.400000	59.65	663.582857	1983.362500	84.785714	2238.342857	16178.782040
21	385.00	36.666667	29.133333	65.80	809.074286	2412.666667	93.571429	2726.047619	21832.163265
22	442.75	40.250000	32.000000	72.25	977.142857	2908.062500	102.785714	3289.142857	29035.102041
23	506.00	44.000000	35.000000	79.00	1170.000000	3476.000000	112.428571	3935.000000	38108.571429
24	575.00	47.916667	38.133333	86.05	1389.960000	4123.229167	122.500000	4671.333333	49420.800000
25	650.00	52.000000	41.400000	93.40	1639.440000	4856.800000	133.000000	5506.200000	63391.680000
26	731.25	56.250000	44.800000	101.05	1920.960000	5684.062500	143.928571	6448.000000	80497.371429
27	819.00	60.666667	48.333333	109.00	2237.142857	6612.666667	155.285714	7505.476190	101275.102041
28	913.50	65.250000	52.000000	117.25	2590.714286	7650.562500	167.071429	8687.714286	126328.163265
29	1015.00	70.000000	55.800000	125.80	2984.502857	8806.000000	179.285714	10004.142857	156331.102041
30	1123.75	74.916667	59.733333	134.65	3421.440000	10087.529167	191.928571	11464.533333	192035.108571

31	1240.00	80.000000	63.800000	143.80	3904.560000	11504.000000	205.000000	13079.000000	234273.600000
32	1364.00	85.250000	68.000000	153.25	4437.000000	13064.562500	218.500000	14858.000000	283968.000000
33	1496.00	90.666667	72.333333	163.00	5022.000000	14778.666667	232.428571	16812.333333	342133.714286
34	1636.25	96.250000	76.800000	173.05	5662.902857	16656.062500	246.785714	18953.142857	409886.302041
35	1785.00	102.000000	81.400000	183.40	6363.154286	18706.800000	261.571429	21291.914286	488447.843200
36	1942.50	107.916667	86.133333	194.05	7126.302857	20941.229167	276.785714	23840.476190	579153.502041
37	2109.00	114.000000	91.000000	205.00	7956.000000	23370.000000	292.428571	26611.000000	683458.285714
38	2284.75	120.250000	96.000000	216.25	8856.000000	26004.062500	308.500000	29616.000000	802944.000000
39	2470.00	126.666667	101.133333	227.80	9830.160000	28854.666667	325.000000	32868.333333	939326.400000
40	2665.00	133.250000	106.400000	239.65	10882.440000	31933.362500	341.928571	36381.200000	1094462.537143
41	2870.00	140.000000	111.800000	251.80	12016.902857	35252.000000	359.285714	40168.142857	1270358.302041
42	3085.25	146.916667	117.333333	264.25	13237.714286	38822.729167	377.071429	44243.047619	1469176.163265
43	3311.00	154.000000	123.000000	277.00	14549.142857	42658.000000	395.285714	48620.142857	1693243.102041
44	3547.50	161.250000	128.800000	290.05	15955.560000	46770.562500	413.928571	53314.000000	1945058.742857
45	3795.00	168.666667	134.733333	303.40	17461.440000	51173.466667	433.000000	58339.533333	2227303.680000
46	4053.75	176.250000	140.800000	317.05	19071.360000	55880.062500	452.500000	63712.000000	2542848.000000
47	4324.00	184.000000	147.000000	331.00	20790.000000	60904.000000	472.428571	69447.000000	2894760.000000
48	4606.00	191.916667	153.333333	345.25	22622.142857	66259.229167	492.785714	75560.476190	3286315.102041
49	4900.00	200.000000	159.800000	359.80	24572.674286	71960.000000	513.571429	82068.714286	3721004.963265
50	5206.25	208.250000	166.400000	374.65	26646.582857	78020.862500	534.785714	88988.342857	4202546.782041
51	5525.00	216.666667	173.133333	389.80	28848.960000	84456.666667	556.428571	96336.333333	4734892.800000
52	5856.50	225.250000	180.000000	405.25	31185.000000	91282.562500	578.500000	104130.000000	5322240.000000
53	6201.00	234.000000	187.000000	421.00	33660.000000	98514.000000	601.000000	112387.000000	5969040.000000
54	6558.75	242.916667	194.133333	437.05	36279.360000	106166.729167	623.928571	121125.333333	6680009.142857
55	6930.00	252.000000	201.400000	453.40	39048.582857	114256.800000	647.285714	130363.342857	7460138.782041
56	7315.00	261.250000	208.800000	470.05	41973.274286	122800.562500	671.071429	140119.714286	8314705.763265
57	7714.00	270.666667	216.333333	487.00	45059.142857	131814.666667	695.285714	150413.476190	9249283.102041
58	8127.25	280.250000	224.000000	504.25	48312.000000	141316.062500	719.928571	161264.000000	10269750.857142
59	8555.00	290.000000	231.800000	521.80	51737.760000	151322.000000	745.000000	172691.000000	11382307.200000
60	8997.50	299.916667	239.733333	539.65	55342.440000	161850.029167	770.500000	184714.533333	12593479.680000

TABLE I—*Continued*

m	S_4	A_{12}	A_{22}	A_{32}	A_{42}	A_{52}	A_{62}	A_{72}	A_{82}
61	9455.00	310.000000	247.800000	557.80	59132.160000	172918.000000	796.428571	197355.000000	13910136.685714
62	9927.75	320.250000	256.000000	576.25	63113.142857	184544.062500	822.785714	210633.142857	15339499.102041
63	10416.00	330.666667	264.333333	595.00	67291.714286	196746.666667	849.571429	224570.047619	16889152.163265
64	10920.00	341.250000	272.800000	614.05	71674.302857	209544.562500	876.785714	239187.142857	18567057.502041
65	11440.00	352.000000	281.400000	633.40	76267.440000	222956.800000	904.428571	254506.200000	20381565.394286
66	11976.25	362.916667	290.133333	653.05	81077.760000	237002.729167	932.500000	270549.333333	22341427.200000
67	12529.00	374.000000	299.000000	673.00	86112.000000	251702.000000	961.000000	287339.000000	24455808.000000
68	13098.50	385.250000	308.000000	693.25	91377.000000	267074.562500	989.928571	304898.000000	26734299.428571
69	13685.00	396.666667	317.133333	713.80	96879.702857	283140.666667	1019.285714	323249.476190	29186932.702041
70	14288.75	408.250000	326.400000	734.65	102627.154286	299920.862500	1049.071429	342416.914286	31824191.843265
71	14910.00	420.000000	335.800000	755.80	108626.502857	317436.000000	1079.285714	362424.142857	34657027.102041
72	15549.00	431.916667	345.333333	777.25	114885.000000	335707.229167	1109.928571	383295.333333	37696868.571429
73	16206.00	444.000000	355.000000	799.00	121410.000000	354756.000000	1141.000000	405055.000000	40955640.000000
74	16881.25	456.250000	364.800000	821.05	128208.960000	374604.062500	1172.500000	427728.000000	44445772.800000
75	17575.00	468.666667	374.733333	843.40	135289.440000	395273.466667	1204.428571	451339.533333	48180220.251429
76	18287.50	481.250000	384.800000	866.05	142659.102857	416786.562500	1236.785714	475915.142857	52172471.902041
77	19019.00	494.000000	395.000000	889.00	150325.714286	439166.000000	1269.571429	501480.714286	56436568.163265
78	19769.75	506.916667	405.333333	912.25	158297.142857	462434.729167	1302.785714	528062.476190	60987115.102041
79	20540.00	520.000000	415.800000	935.80	166581.360000	486616.000000	1336.428571	555687.000000	65839299.428571
80	21330.00	533.250000	426.400000	959.65	175186.440000	511733.362500	1370.500000	584381.200000	71008903.680000
81	22140.00	546.666667	437.133333	983.80	184120.560000	537810.666667	1405.000000	614172.333333	76512321.600000
82	22970.25	560.250000	448.000000	1008.25	193392.000000	564872.062500	1439.928571	645088.000000	82366573.714286
83	23821.00	574.000000	459.000000	1033.00	203009.142857	592942.000000	1475.285714	677156.142857	88589323.102041
84	24692.50	587.916667	470.133333	1058.05	212980.474286	622045.229167	1511.071429	710405.047619	95198891.363265
85	25585.00	602.000000	481.400000	1083.40	223314.582857	652206.800000	1547.285714	744863.342857	102214274.780204

86	26498.75	616.250000	492.800000	1109.05	234020.160000	683452.062500	1583.928571	780560.000000	109655160.685714
87	27434.00	630.666667	504.333333	1135.00	245106.000000	715806.666667	1621.000000	817524.333333	117541944.000000
88	28391.00	645.250000	516.000000	1161.25	256581.000000	749296.562500	1658.500000	855786.000000	125895744.000000
89	29370.00	660.000000	527.800000	1187.80	268454.160000	783948.000000	1696.428571	895375.000000	134738421.257143
90	30371.25	674.916667	539.733333	1214.65	280734.582857	819787.529167	1734.785714	936321.676190	144092594.780204
91	31395.00	690.000000	551.800000	1241.80	293431.474286	856842.000000	1773.571429	978656.714286	153981659.363265
92	32441.50	705.250000	564.000000	1269.25	306554.142857	895138.562500	1812.785714	1022411.142857	164429803.102041
93	33511.00	720.666667	576.333333	1297.00	320112.000000	934704.666667	1852.428571	1067616.333333	175462025.142857
94	34603.75	736.250000	588.800000	1325.05	334114.560000	975568.062500	1892.500000	1114304.000000	187104153.600000
95	35720.00	752.000000	601.400000	1353.40	348571.440000	1017756.800000	1933.000000	1162506.200000	199382863.680000
96	36860.00	767.916667	614.133333	1382.05	363492.360000	1061299.229167	1973.928571	1212255.333333	212325696.000000
97	38024.00	784.000000	627.000000	1411.00	378887.142857	1106224.000000	2015.285714	1263584.142857	225961075.102041
98	39212.25	800.250000	640.000000	1440.25	394765.714286	1152560.062500	2057.071429	1316525.714286	240318328.163265
99	40425.00	816.666667	653.133333	1469.80	411138.102857	1200336.666667	2099.285714	1371113.476190	255427703.902041
100	41662.50	833.250000	666.400000	1499.65	428014.440000	1249583.362500	2141.928571	1427381.200000	271320391.680000

where

$$\bar{x} = \frac{1}{m}\sum_{j=1}^{m} x_j, \qquad \bar{y} = \frac{1}{m}\sum_{j=1}^{m} y_j. \tag{3.3}$$

The transformation on $\{y_j\}$ is not necessary but convenient and a change in formula (3.18) is necessary if it is not made.

The residual equations on the transformed data are

$$R_j = \sum_{i=0}^{n} b_i X_j^i - X_j, \qquad (j = 1, \cdots, m) \tag{3.4}$$

and the sum of the squares of the residuals is

$$\sum_{j=1}^{m} R_j^2 = \sum_{j=1}^{m}\left[\sum_{i=0}^{n} b_i X_j^i - Y_j\right]^2 = f(b_0, \cdots, b_n). \tag{3.5}$$

The condition that this function be a minimum yields the following system of equations

$$\frac{\partial f(b_0, \cdots, b_n)}{\partial b_k} = 2\sum_{j=1}^{m}\left[\sum_{i=0}^{n} b_i X_j^i - Y_j\right] X_j^k = 0, \quad (k = 0, \cdots, n). \tag{3.6}$$

Rearranging equations (3.6) to obtain the coefficients of the unknowns b_i, it is easily seen that we have the following system of $n + 1$ equations: [All summations going from 1 to m]

$$\begin{cases} mb_0 + \Sigma X_j b_1 + \Sigma X_j^2 b_2 + \cdots + \Sigma X_j^n b_n = \Sigma Y_j \\ \Sigma X_j b_0 + \Sigma X_j^2 b_1 + \Sigma X_j^3 b_2 + \cdots + \Sigma X_j^{n+1} b_n = \Sigma X_j Y_j \\ \Sigma X_j^2 b_0 + \Sigma X_j^3 b_1 + \Sigma X_j^4 b_2 + \cdots + \Sigma X_j^{n+2} b_n = \Sigma X_j^2 Y_j \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \Sigma X_j^n b_0 + \Sigma X_j^{n+1} b_1 + \Sigma X_j^{n+2} b_2 + \cdots + \Sigma X_j^{2n} b_n = \Sigma X_j^n Y_j. \end{cases} \tag{3.7}$$

In order to express the coefficients of b_i in a desired form it becomes necessary to consider two cases; namely, where m is odd and where m is even.

Case I. $m = 2r - 1$. In this case we have

$$\bar{x} = \frac{1}{2r-1}\sum_{j=1}^{2r-1} x_j = \frac{1}{2r-1}\left[(2r-1)x_1 + h\sum_{k=1}^{2r-2} k\right] = x_1 + (r-1)h \tag{3.8}$$

so that by (3.1) and (3.2)

$$X_j = x_1 + (j-1)h - x_1 - (r-1)h = (j-r)h \tag{3.9}$$

and

$$\sum_{j=1}^{2r-1} X_j^i = \begin{cases} 0 \text{ if } i \text{ is odd;} \\ 2h^i \sum_{k=1}^{r-1} k^i \text{ if } i \text{ is even.} \end{cases} \tag{3.10}$$

Case II. $m = 2r$. For an even number of observations we have

$$\bar{x} = \frac{1}{2r}\sum_{j=1}^{2r} x_j = \frac{1}{2r}\left[2rx_1 + h\sum_{k=1}^{2r-1} k\right] = x_1 + \tfrac{1}{2}(2r-1)h \tag{3.11}$$

so that by (3.1) and (3.2) we have

$$X_j = x_1 + (j-1)h - x_1 - \tfrac{1}{2}(2r-1)h = \tfrac{1}{2}(2j - 2r - 1)h \tag{3.12}$$

and

$$\sum_{j=1}^{2r} X_j^i = \begin{cases} 0 \text{ if } i \text{ is odd;} \\ (\tfrac{1}{2})^i 2h^i \sum_{k=1}^{r} (2k-1)^i \text{ if } i \text{ is even.} \end{cases} \tag{3.13}$$

It is now convenient to define

$$S_i = \begin{cases} \sum_{k=1}^{r-1} k^i \text{ if } m \text{ is odd;} \\ (\tfrac{1}{2})^i \sum_{k=1}^{r} (2k-1)^i \text{ if } m \text{ is even.} \end{cases} \qquad [i = 2, 4, 6, \cdots, 2n]. \tag{3.14}$$

The system (3.7) may now be written schematically as follows

$$\left\{\begin{array}{cccccc|c} b_0 & b_1 & b_2 & b_3 & \cdots & b_n & c \\ \hline m & 0 & 2S_2h^2 & 0 & \cdot & 2S_nh^n & \Sigma Y_j \\ 0 & 2S_2h^2 & 0 & 2S_4h^4 & \cdot & 2S_{n+1}h^{n+1} & \Sigma X_jY_j \\ 2S_2h^2 & 0 & 2S_4h^4 & 0 & \cdot & 2S_{n+2}h^{n+2} & \Sigma X_j^2Y_j \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2S_nh^n & 2S_{n+1}h^{n+1} & 2S_{n+2}h^{n+2} & \cdot & \cdot & 2S_{2n}h^{2n} & \Sigma X_j^nY_j \end{array}\right. \tag{3.15}$$

where in the last row and last column of b_n, $S_{n+q} = 0$ if $n + q$ is odd, $(q = 0, 1, \cdots, n)$. (3.15) is now transformed by dividing *each* equation by 2 and *successive* equations by $h^0, h, h^2, \cdots, h^n$, respectively; the result is easily obtained. The new system thus obtained is further transformed by replacing b_i by $\alpha_i h^{-i}$ and we finally have a system of equations in α_i

$$\left\{\begin{array}{ccccccc|c} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & & \alpha_n & c \\ \hline \tfrac{1}{2}m & 0 & S_2 & 0 & \cdots & S_n & \tfrac{1}{2}\Sigma Y_j \\ 0 & S_2 & 0 & S_4 & \cdots & S_{n+1} & \tfrac{1}{2}h^{-1}\Sigma X_jY_j \\ S_2 & 0 & S_4 & 0 & \cdots & S_{n+2} & \tfrac{1}{2}h^{-2}\Sigma X_j^2Y_j \\ 0 & S_4 & 0 & S_6 & \cdots & S_{n+3} & \tfrac{1}{2}h^{-3}\Sigma X_j^3Y_j \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ S_n & S_{n+1} & S_{n+2} & S_{n+3} & \cdots & S_{2n} & \tfrac{1}{2}h^{-n}\Sigma X_j^nY_j \end{array}\right. \tag{3.16}$$

where $S_i = 0$ if i is an odd number.

Before proceeding with the solution of this system of equations it has been

found convenient to first divide each equation, except the first, by S_2. Letting $S_i' = S_iS_2^{-1}$, we finally have

$$
(3.17)\qquad
\begin{array}{c|ccccccc}
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & & \alpha_n & c \\
\hline
\tfrac{1}{2}m & 0 & S_2 & 0 & \cdots & S_n & c_1 \\
0 & 1 & 0 & S_4' & \cdots & S_{n+1}' & c_2 \\
1 & 0 & S_4' & 0 & \cdots & S_{n+2}' & c_3 \\
0 & S_4' & 0 & S_6' & \cdots & S_{n+3}' & c_4 \\
\cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\
S_n' & S_{n+1}' & S_{n+2}' & S_{n+3}' & \cdots & S_{2n}' & c_{n+1}
\end{array}
$$

where again $S_i' = S_i = 0$ if i is an odd number and

$$
(3.18)\qquad
\begin{cases}
c_1 = \dfrac{1}{2}\displaystyle\sum_{j=1}^{m} Y_j = \frac{1}{2}\left[\sum_{j=1}^{m} y_j - m\left[\frac{1}{m}\right]\sum_{j=1}^{m} y_j\right] = 0 \\
c_i = \tfrac{1}{2}S_2^{-1}h^{-i+1}\displaystyle\sum_{j=1}^{m} X_j^{i-1}Y_j, \qquad [i = 2, \cdots, n+1].
\end{cases}
$$

4. The Derived Matrix. The system of $n+1$ linear equations in the $n+1$ unknowns $\alpha_i (i = 0, 1, \cdots, n)$, represented schematically by (3.17), may now be solved by using the Crout algorithm. This algorithm forms a *derived matrix*[7] from the matrix (3.17) of the general form

$$
(4.1)\qquad
\begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1,n+1} & K_1 \\
A_{21} & A_{22} & & A_{2,n+1} & K_2 \\
A_{31} & A_{32} & & A_{3,n+1} & K_3 \\
\cdot & \cdot & & \cdot & \cdot \\
A_{n+1,1} & A_{n+1,2} & & A_{n+1,n+1} & K_{n+1}
\end{pmatrix}.
$$

The elements of the *derived determinant*, $\Delta = |A_{ij}|$, are functions of the sums S_i', S_i, $(i = 2, 4, \cdots, 2n)$, and $\frac{1}{2}m$. The sums S_i' and S_i in turn may be computed by the well known formulas[8]

Case I, $(m = 2r - 1)$,

$$
(4.2)\qquad
\sum_{i=1}^{r} k^i = \frac{1}{i+1}r^{i+1} + \frac{1}{2}r^i + \frac{1}{2}\binom{i}{1}B_1 r^{i-1} - \frac{1}{4}\binom{i}{3}B_3 r^{i-3} + \frac{1}{6}\binom{i}{5}B_5 r^{i-5} - \cdots
$$

and Case II, $(m = 2r)$,

[7] Crout called this the *auxiliary* matrix.

[8] Adams, E. P., "Smithsonian Mathematical Formulae and Tables of Elliptic Functions", Smithsonian Institution, Washington, D. C., (1939), p. 27.

$$\left(\frac{1}{2}\right)^i \sum_{k=1}^{r} (2k-1)^i = \left(\frac{1}{2}\right)^i \left[\sum_{k=1}^{2r} k^i - 2^i \sum_{k=1}^{r} k^i\right], \tag{4.3}$$

where B_s $(s = 1, 2, \cdots)$ are the Bernoulli numbers. Thus these sums may be expressed in terms of r only. Consequently, the elements A_{ij} *depend only upon the number of observations* and thus may be computed once and for all and tabulated for a range of observation; (see Table I). The size of such a table is greatly reduced by the fact, that the derived matrix obtained from (3.17) is such that the elements $A_{ij} = 0$ if $i + j$ is an odd number. It is further reduced by the fact that the formulas which give the solutions for α_i do not employ all of the non-zero A_{ij}; (see §5).

The column of constant terms K_i $(i = 1, \cdots, n+1)$ is expressed in terms of c_1 [see (3.18)] and A_{ij}. Thus it is possible to express the solutions for α_i in simple closed formulas in terms of K_i and A_{ij}. There remains then only to reverse the transformation to obtain a_i, $(i = 0, \cdots, n)$.

5. The Formulas. The formulas expressing A_{ij} in terms of r are obtained by first evaluating the sums S_i and S_i' by (4.2) and (4.3), whichever applies, and then applying the algorithm to (3.17). This has been done for the polynomials up to and including polynomials of the sixth degree. The formulas which were used to compute Table I; that is, for polynomials of degree ≤ 4, are exhibited below. Considerable elementary algebraic manipulation which was necessary will not be displayed.

(5.1)

A_{ij}	Case I. $(m = 2r - 1)$	Case II. $(m = 2r)$
A_{11}	$\frac{1}{2}(2r-1)$	r
A_{12}	$\frac{1}{3}(r-1)r$	$\frac{1}{12}(4r^2-1)$
A_{13}	$A_{12}A_{24}$	$A_{12}A_{24}$
A_{22}	1	1
A_{24}	$\frac{1}{3}(3r^2-3r-1)$	$\frac{1}{30}(12r^2-7)$
A_{33}	$\frac{1}{15}(2r+1)(2r-3)$	$\frac{4}{15}(r^2-1)$
A_{35}	$\frac{1}{7}(6r^2-6r-5)$	$\frac{1}{15}(12r^2-13)$
A_{44}	$\frac{8}{315}(r+1)(r-2)\,A_{33}$	$\frac{8}{140}(4r^2-9)\,A_{33}$
A_{53}	$A_{35}A_{33}$	$A_{35}A_{33}$
A_{55}	$\frac{4}{63}(2r+3)(2r-5)\,A_{44}$	$\frac{16}{63}(r^2-4)\,A_{44}$

and

$$A_{31} = A_{22} \quad \text{and} \quad A_{42} = A_{51} = A_{24}. \tag{5.2}$$

Remark: We note that the formulas for the A_{ij} may make an interesting algebraic study. The Crout algorithm yields the following general formulas for these elements:

$$j = i: \quad A_{ii} = S'_{i-2} - \sum_{w=1}^{i-1} A_{iw}A_{wi};$$

$$j > i: \quad A_{ij} = \left[S'_{i+j-2} - \sum_{w=1}^{i-1} A_{iw}A_{wj}\right] \Big/ A_{ii};$$

$$j < i: A_{ij} = A_{ji}A_{jj}.$$

We are lead to the following *conjecture*

$$A_{i+1,i+1} = \frac{(i/2)^2}{(2i-1)(2i+1)}[m^2 - i^2]A_{ii}.$$

The elements of the column of constant terms are given by the general formula

$$K_i = \left[c_i - \sum_{j=1}^{i-1} K_j A_{ij}\right] \Big/ A_{ii} \qquad (i = 1, \cdots, n+1) \tag{5.3}$$

where $A_{ij} = 0$ if $i + j$ is an odd number. For the case under consideration, namely, $n \leq 4$, we have

$$\begin{cases} K_1 = 0, \quad K_2 = c_2, \quad K_3 = c_3/A_{33} \\ K_4 = [c_4 - K_2A_{24}]/A_{44} \text{ since } A_{42} = A_{24}, \quad K_5 = [c_5 - K_3A_{35}]/A_{55}. \end{cases} \tag{5.4}$$

The general formula for α_i is

$$\alpha_i = K_{i+1} - \sum_{j=i+2}^{n} \alpha_j A_{i+1,j+1} \qquad [i = 0, \cdots, n], \tag{5.5}$$

where again $A_{ij} = 0$ if $i + j$ is an odd number. For the case under consideration we may tabulate these formulas as follows:

(5.6)

α_i	n = 1	n = 2	n = 3	n = 4
α_0	0	$-\alpha_2A_{13}$	$-\alpha_2A_{13}$	$-\alpha_2A_{13} - \alpha_4A_{15}$
α_1	K_2	K_2	$K_2 - \alpha_3A_{24}$	$K_2 - \alpha_3A_{24}$
α_2		K_3	K_3	$K_3 - \alpha_4A_{35}$
α_3			K_4	K_4
α_4				K_5

To obtain b_i $(i = 0, \cdots, n)$ reverse the transformation; i.e.,

$$b_i = \alpha_i h^{-i} \qquad [i = 0, \cdots, n]. \tag{5.7}$$

a'_0 and a_i $(i = 1, \cdots, n)$ is easily obtained from b_i by a Horner's Method procedure of synthetic division using $-\bar{x}$ as a divisor. Finally,

$$a_0 = a'_0 + \bar{y}. \tag{5.8}$$

6. Summary. The procedure for fitting a curve of desired degree to a given set of data by the above method may be outlined as follows:

1. Transform the data by (3.2).
2. Choose the appropriate set of A_{ij} and S_2 from Table I.[9]
3. Compute c_i by (3.18).
4. Compute K_i by (5.3).
5. Compute α_i by (5.5); $[i = n, \cdots, 0]$.
6. Compute b_i by (5.7).
7. Compute a_0' and a_i $(i = 1, \cdots, n)$ by synthetic division.
8. Compute a_0 by (5.8).

7. Examples.

No. 1. Fit a cubic to the data

x	1	2	3	4	5	6	7	8	9	10	11
y	108	56	0	−54	−100	−132	−144	−130	−84	0	128

Solution:

$$\bar{x} = 6; \qquad \bar{y} = -32; \qquad h = 1; \qquad m = 11.$$

X	−5	−4	−3	−2	−1	0	1	2	3	4	5
Y	140	88	32	−22	−68	−100	−112	−98	−52	32	160

From Table I: $S_2 = 55$; $A_{12} = 10$; $A_{24} = 17.8$; $A_{23} = 7.8$; $A_{44} = 56.16$.

From (3.18)	From (5.4)	From (5.6)	a_i
$c_1 = 0$	$K_1 = 0$	$\alpha_3 = 1$	$a_3 = 1$
$c_2 = -5.2$	$K_2 = -5.2$	$\alpha_2 = 10$	$a_2 = -8$
$c_3 = 78$	$K_3 = 10$	$\alpha_1 = -23$	$a_1 = -35$
$c_4 = -36.4$	$K_4 = 1$	$\alpha_0 = -100$	$a_0 = 150$

$\therefore y = x^3 - 8x^2 - 35x + 150.$

[9] If Table I does not contain the appropriate A_{ij}, they may be computed from (5.1) and S_2 from (3.14). If a polynomial of degree higher than four is desired, the formulas for A_{ij} must be extended beyond (5.1) and is accomplished by applying Crout's algorithm to (3.17) and computing the sums by (4.2) or (4.3). See also our remark in §5.

No. 2. Fit a fourth degree equation to the data (x_i, y_i).

Given:

x	y
0	.5179
.2	.7049
.4	.9625
.6	1.3090
.8	1.7650
1.0	2.3530
1.2	3.0990
1.4	4.0300
1.6	5.1750
1.8	6.5660
2.0	8.2380
2.2	10.2260
2.4	12.5690
2.6	15.3080
2.8	18.4870
3.0	22.1510
3.2	26.3467

Solution:

$m = 17$
$\bar{x} = 1.6$
$\bar{y} = 8.224$
$h = .2$
$S_2 = 204$
$2S_2 = 408$

X	Sum Y	Diff. Y
0	−3.0490	−3.0490
.2	−5.8520	2.5360
.4	−5.1110	5.1390
.6	−3.8690	7.8730
.8	−2.1140	10.8040
1.0	.1690	13.9990
1.2	3.0015	17.5245
1.4	6.4079	21.4461
1.6	10.4166	25.8288
Check: 0		

Note: The arrangement of the data in columns of sums and differences of Y about the x mean expidites the computation of c_i. Thus when
$i - 1$ is odd use Diff. Y column
$i - 1$ is even use Sum Y column.

i	h^{-i+1}	c_i	K_i	α_{i-1}	b_{i-1}
1	1	0	0	−3.048999	−3.048999
2	5	1.498883	1.498883	1.262598	6.312990
3	25	2.446048	.128739	.122987	3.074675
4	125	66.331372	.005495	.005495	.686875
5	625	149.767474	$.942937 \times 10^{-4}$	$.942937 \times 10^{-4}$	.058933

.058933	.686875	3.074675	6.312990	5.175001	−1.6
	−.094293	−.948131	−3.402470	−4.656832	
.058933	.592582	2.126544	2.910520	.518169	
	−.094293	−.797262	−2.126851		
.058933	.498289	1.329282	.783669		
	−.094293	−.646394			
.058933	.403996	.682888			
	−.094293				
.058933	.309703				

$$.0589x^4 + .3097x^3 + .6829x^2 + .7837x + .5182 = y.$$

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SCHWARZ' INEQUALITY AND THE METHODS OF RAYLEIGH-RITZ AND TREFFTZ

By J. B. Diaz and Alexander Weinstein

The main purpose of the present note is to establish that both the Rayleigh-Ritz method and Trefftz procedure can be derived, in the case of quadratic functionals, by a simple and direct application of Schwarz' inequality and Green's formula. Our approach is shorter and more elementary than the analysis given by Friedrichs[1], who showed the connection of Trefftz' method with Legendre's transformation.

1. Schwarz' inequality and Green's formula. For the sake of definiteness, we take as our quadratic functional the Dirichlet integral

$$D(\varphi) = \int\int_R (\varphi_x^2 + \varphi_y^2)\, dx\, dy, \tag{1}$$

where R is a domain in the plane. (The number of independent variables is irrelevant for our discussion.) Putting

$$D(\varphi, \psi) = \int\int_R (\varphi_x \psi_x + \varphi_y \psi_y)\, dx\, dy, \tag{2}$$

we have Schwarz' inequality

$$[D(\varphi, \psi)]^2 \leqq D(\varphi) \cdot D(\psi). \tag{3}$$

Besides this inequality we shall employ the classical Green's formula, which is constantly used in the methods of Rayleigh-Ritz and Trefftz

$$\int\int_R \varphi \cdot \Delta\psi\, dx\, dy + D(\varphi, \psi) = \int_C \varphi(\partial\psi/\partial n)\, ds, \tag{4}$$

$\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ being the Laplacian, C the boundary of R, and n the outer normal to C.

2. Dirichlet's problem and Dirichlet's principle. Let us consider the Dirichlet problem: to determine $u(x, y)$ in R, satisfying

$$\Delta u = 0, \qquad \text{in } R, \tag{5}$$

and

$$u = \text{a given function, on } C. \tag{6}$$

The Rayleigh-Ritz method, which is essentially nothing else but Dirichlet's principle, furnishes an upper bound for $D(u)$. Let $w(x, y)$ take the same values as $u(x, y)$ on the boundary, that is

$$w = u, \qquad \text{on } C; \tag{7}$$

[1] Kurt Friedrichs, "Ein Verfahren der Variationsrechnung das Minimum eines Integral als das Maximum eines anderen Ausdruckes darzustellen, Gottingen Nachrichten, 1929, pp. 13-20.

then we have, as known

$$D(u) \leqq D(w). \tag{8}$$

Let us show that this classical result can be obtained starting from Schwarz' inequality. We have

$$[D(u, w)]^2 \leqq D(u) \cdot D(w), \tag{9}$$

and by Green's theorem

$$D(u, w) = D(u, u + (w - u)) = D(u), \tag{10}$$

since $w - u$ vanishes on C and u satisfies (5). Thus

$$[D(u)]^2 \leqq D(u) \cdot D(w) \tag{11}$$

from which (8) follows, the trivial case $D(u) = 0$ being excluded.

3. Trefftz' method. Now we are going to show that the method of Trefftz[2], which furnishes a lower bound for $D(u)$, can, also be easily deduced from Schwarz' inequality. Let $v(x, y)$ be a non-constant function satisfying the equation

$$\Delta v = 0, \qquad \text{in } R; \tag{12}$$

then, according to Trefftz, we have

$$D(\lambda v) \leqq D(u), \tag{13}$$

λ being a suitably chosen real constant (see eq. (17)). In order to obtain (13) we proceed as in the preceding section. We have

$$[D(u, v)]^2 \leqq D(u) \cdot D(v), \tag{14}$$

and by Green's theorem

$$D(u, v) = -\int\int_R u \cdot \Delta v \, dx \, dy + \int_C u \frac{\partial v}{\partial n} \, ds = \int_C u \frac{\partial v}{\partial n} \, ds. \tag{15}$$

Thus

$$\frac{\left[\int_C u(\partial v/\partial n) \, ds\right]^2}{D(v)} \leqq D(u). \tag{16}$$

Since the left hand side of (16) is known, we have here essentially the lower bound for $D(u)$ obtained by Trefftz. If we set

$$\lambda = \frac{\int_C u(\partial v/\partial n) \, ds}{D(v)}, \tag{17}$$

we obtain (13), which is identical with the result formulated by Trefftz.

[2] E. Trefftz, Ein Gegenstück zum Ritzschen Verfahren, Verhandlungen, Congress für Technische Mechanik, Zürich, 1927, p. 131.

4. Neumann's problem. The technique outlined above may be applied to other boundary value problems, and to some other cases provided that the appropriate form of Schwarz' inequality is known. As an example we consider Neumann's problem[3].

Let $u(x, y)$ be the solution of the boundary value problem

$$\Delta u = 0, \qquad \text{in } R, \tag{18}$$

(19) $\partial u/\partial n =$ a given function, on C, the mean value of the given function being zero on C.

A lower bound for $D(u)$ may be obtained as follows: Let $v(x, y)$ be any non-constant function. From Schwarz' inequality

$$[D(u, v)]^2 \leqq D(u) \cdot D(v), \tag{20}$$

and from Green's formula

$$D(u, v) = \int_C v(\partial u/\partial n)\, ds. \tag{21}$$

It follows that

$$\frac{\left[\int_C v(\partial u/\partial n)\, ds\right]^2}{D(v)} \leqq D(u), \tag{22}$$

which yields a lower bound for $D(u)$.

In order to obtain an upper bound for $D(u)$, we start with the following Schwarz' inequality

$$\left[\int\int_R (\varphi_1 \psi_1 + \varphi_2 \psi_2)\, dx\, dy\right]^2 \leqq \int\int_R (\varphi_1^2 + \varphi_2^2)\, dx\, dy \cdot \int\int_R (\psi_1^2 + \psi_2^2)\, dx\, dy \tag{23}$$

Let $w_1(x, y)$ and $w_2(x, y)$ be two functions satisfying the conditions

$$\partial w_1/\partial x + \partial w_2/\partial y = 0, \qquad \text{in R}, \tag{24}$$

and

$$w_1 n_x + w_2 n_y = \partial u/\partial n, \qquad \text{on } C, \tag{25}$$

n_x and n_y being the components of the outer normal to C. We have, from (23)

$$\left[\int\int_R u_x w_1 + u_y w_2)\, dx\, dy\right]^2 = D(u) \cdot \int\int_R (w_1^2 + w_2^2)\, dx\, dy, \tag{26}$$

and by Green's theorem, using (24) and (25)

$$\int\int_R (u_x\, w_1 + u_y w_2)\, dx\, dy = D(u). \tag{27}$$

Thus[4]

[3] For a discussion of Neumann's problem see a forthcoming paper by J. B. Diaz and Alexander Weinstein, The Torsional Rigidity and Variational Methods.

[4] The inequality (26) is nothing else but Kelvin's theorem. See, e.g., H. Lamb, Hydrodynamics, Sixth Edition, New York, 1945, p. 47.

$$D(u) \leqq \int\int_R (w_1^2 + w_2^2)\, dx\, dy, \tag{28}$$

the trivial case $D(u) = 0$ being excluded.

5. Concluding remarks. As is well known, the progress achieved by the Rayleigh-Ritz method, as compared to the usual Dirichlet's principle, consists in adding to the function w which appears in (7) a linear combination of functions which vanish on C, and determining the arbitrary constants so as to improve the upper bound for $D(u)$. A similar procedure has been used by Trefftz, leading to an improvement of his lower bound. It should be noted that this method of "arbitrary parameters" is nothing else than the familiar procedure of orthogonal projection in a Hilbert space with the metric being given by the Dirichlet integral.

Using the language of Hilbert space, in which a function is called a vector, and the metric is again defined by the Dirichlet integral, we may say in conclusion that the Rayleigh-Ritz method (see section 2) consists in considering the unknown function (or vector) u as a side of a right triangle of which w is the hypotenuse, whereas in Trefftz' method (section 3), u is taken as the hypotenuse of a right triangle having $\lambda\, v$ [see (13)], as a side.

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ON THE FLOW PATTERNS COMMON TO CERTAIN CLASSES OF PLANE FLUID MOTIONS

By D. Gilbarg

1. Introduction. A fluid motion is generally described by giving both the flow pattern (i.e. the streamlines) and the magnitude of the flow velocity at every point. Although the flow pattern and the velocity are two distinct features of the same flow they are clearly interdependent, but it is not clear to what extent. This situation suggests the following general problem, the solution to which would show how far the geometry of a flow determines also the dynamics of the motion:

I. Given any fluid motion; what is the set of all dynamically distinct flows which have the same streamlines as the given flow?

A corollary problem whose solution would give some insight into the role of compressibility, viscosity, etc., in determining the geometry of the corresponding types of fluid motions is the following:

II. What are all flow patterns which two different classes of flows (e.g. compressible and incompressible potential flows) have in common?

As far as the author knows, problems of type I have not been previously studied. However, an interesting example of problem II was solved by Hamel[1], who shows that the most general flow patterns which are common to both the steady plane potential flows of an ideal incompressible fluid and the steady plane flows of a viscous incompressible fluid are those of the spiral flows which are obtained by superposition of circular vortex and radial potential flows.

This paper will be concerned only with steady plane flows of non-viscous fluids, and therefore the word 'flow' will be used here only in this sense.

Problem I is solved here for incompressible flows; that is, all incompressible flows are determined which have the same streamlines as an arbitrary given flow of an incompressible fluid. It is shown that all such geometrically identical flows have strictly proportional velocity vector fields (and are therefore essentially the same dynamically), except for those flows having constant velocity on streamlines. A simple application of this result will determine all those flows which at the same time have the flow patterns of possible potential flows of an incompressible fluid.

Problem II is solved here for the classes of potential gas flows and potential flows of incompressible fluids. These two classes of motions will be seen to have in common only the flow patterns corresponding to circular vortex and radial (source or sink) flows.

2. Incompressible flows. Let a steady plane flow of an incompressible non-viscous fluid be defined over some plane region by a stream function $\psi(x, y)$. Necessary and sufficient that another such flow with stream function $\Psi(x, y)$

[1] G. Hamel, Spiralförmige Bewegungen Zäher Flüssigkeiten, Jahresbericht der Deutschen Mathematiker Vereinigung **25**, 1916–17, p. 34.

have the same streamlines is that a functional relation, say $\Psi = f(\psi)$, exist between the stream functions. We inquire after all such possible flows $f(\psi)$, assuming that f is suitably differentiable.

The equations of motion for the two flows in terms of their stream functions ψ, Ψ are

$$\text{(1a)} \qquad \psi_y\psi_{xy} - \psi_x\psi_{yy} = p_x/\rho, \qquad -\psi_y\psi_{xx} + \psi_x\psi_{xy} = -p_y/\rho$$

$$\text{(1b)} \qquad \Psi_y\Psi_{xy} - \Psi_x\Psi_{yy} = -\bar{p}_x/\bar{\rho}, \qquad -\Psi_y\Psi_{xx} + \Psi_x\Psi_{xy} = -\bar{p}_y/\bar{\rho}$$

where p, $\bar{p}$ and ρ, $\bar{\rho}$ are the respective fluid pressures and (constant) densities. Replacing Ψ by $f(\psi)$ in (1b) and using (1a), we have, as is readily seen,

$$f'^2 p_x/\rho = \bar{p}_x/\bar{\rho}, \text{ and } f'^2 p_y/\rho = \bar{p}_y/\bar{\rho},$$

where the primes signify differentiation with respect to ψ. The left hand sides must satisfy the compatibility condition $\bar{p}_{xy} = \bar{p}_{yx}$, and therefore,

$$f'f''(p_x\psi_y - p_y\psi_x) = 0.$$

Since, by (1a),

$$\psi_x\partial/\partial x(\psi_x^2 + \psi_y^2) - \psi_x\partial/\partial y(\psi_x^2 + \psi_y^2) = 2(p_y\psi_x - p_x\psi_y)/\rho$$

the preceding equation can be written in the form,

$$\text{(2)} \qquad f'f''(q_x\psi_y - q_y\psi_x) = 0,$$

where $q = \sqrt{\psi_x^2 + \psi_y^2}$ is the magnitude of the velocity.

It follows that at every point in the flow plane either (a) $f'f'' = 0$, or (b) $q_x\psi_y - q_y\psi_x = 0$. Since the flow ψ is given, the set of points for which the latter condition holds is known. Consider the open connected components in which condition (b) is satisfied and the components of the complement, in which, by the continuity of $f'f''$, (a) must hold.[2]

In those components where (a) holds, it follows that $\Psi = a\psi + b$, where a and b are constants. Condition (b) on the other hand, imposes no condition on f, which may therefore be chosen arbitrarily, but implies that the given flow must be such that the velocity is constant on streamlines. These are the flows for which $q^2 = \psi_x^2 + \psi_y^2 = g(\psi)$ for some function g. Thus the stream function Ψ of any flow which has the same streamlines as that given by ψ must be of the form $\Psi = a\psi + b$, with fixed a, b, in every connected flow region throughout which the magnitude of the velocity is not constant on streamlines, and can be an arbitrary function of ψ in any flow region where the velocity is constant on streamlines.[3] And conversely, it is seen at once from the equations of motion

[2] If the point set B where condition (b) holds contains no inner points, then, by the continuity of $f'f''$, the (closed) set A in which condition (a) prevails is the entire flow plane. Also, by the continuity of $q_x\psi_y - q_y\psi_x$, if the (closed) set B is not the entire flow plane, set A contains open connected components.

[3] More precise determinations of Ψ might be obtained by investigating the exact nature of those flows ψ for which $q = q(\psi)$ holds either in the whole or in part of the flow.

(1a) and (1b) that functions Ψ as thus defined are stream functions of an incompressible flow having the same streamlines as the flow with stream function ψ.

Hence we can state the following result:

Let any steady plane flow of an incompressible non-viscous fluid be given by the stream function $\psi(x, y)$. If the motion in any region D is such that the velocity is constant on streamlines, i.e. $\psi_x^2 + \psi_y^2 = g(\psi)$ for some function g, then all other flows having the same streamlines (in D) are given (in D) by stream functions $\Psi = f(\psi)$ where f is arbitrary. If ψ is not of this type, then all other flows Ψ having the same streamlines are of the form $\Psi = a\psi + b$ where a and b are arbitrary real constants, and therefore have velocity fields which are proportional to that of the given flow.

An application of this result can be made to determine those rotational flows which have the same flow patterns as potential flows of an incompressible fluid. This is the limiting case of viscosity zero of the more general problem considered by Hamel[4], in which rotational viscous flows are compared to potential flows.

Let ψ be the stream function of a potential flow. Then all other potential flows $\Psi = f(\psi)$ having the same streamlines must be of the form $\Psi = a\psi + b$. For, $0 = \Psi_{xx} + \Psi_{yy} = f''(\psi_x^2 + \psi_y^2) + f'(\psi_{xx} + \psi_{yy}) = f''(\psi_x^2 + \psi_y^2)$, and therefore f is linear in ψ. Conversely, if f is linear, Ψ is obviously harmonic. Hence a flow $\Psi = f(\psi)$ can be rotational only if f is non-linear, and these Ψ give all such flows with the same streamlines as ψ. However, by the preceding general result, the only cases when the two flows Ψ and ψ are connected by a non-linear relation, are those for which ψ is of the form $\psi = \psi(q)$, and in this case the relation can be arbitrary.

To find the potential flows of the form $\psi = \psi(q)$, we observe by the same argument as above that, since ψ and $\log q$ are both harmonic functions, $\log q = a\psi + b$, with a and b constants. Let $w = \varphi + i\psi$ be the complex velocity potential of the flow, then since

$$i \log dw/dz = \theta + i \log q$$

(where θ is the flow direction with respect to the x-axis), θ and $\log q$, and thus θ and $a\psi$ must be harmonic conjugates; it follows that $\theta = a\varphi$ to within an additive constant. As a consequence, $i \log dw/dz = a(w - w_0)$, which, when integrated, generally defines a flow with concentric circles as streamlines, corresponding to a vortex flow. When $a = 0$, the streamlines degenerate to parallel straight lines and the flow to a uniform flow. Hence the only flow patterns common to both the rotational and irrotational flows of an incompressible fluid are those of the family of circular vortex flows, including the special case of the parallel flows.

3. Flow Patterns of Compressible and Incompressible Potential Flows. To obtain an indication of the effect of compressibility in distinguishing the flow patterns of compressible and incompressible fluids, we shall determine all

[4] Cf. Introduction and footnote 1.

flow patterns common to the potential flows of both incompressible fluids and gases.

Let $\varphi(x, y)$, $\psi(x, y)$ be the potential and stream functions respectively of a potential flow of an incompressible fluid, and $\Phi(x, y)$, $\Psi(x, y)$ the potential and stream function of an irrotational gas flow having the same streamlines, (and therefore the same equipotential lines), as the former flow. Then $\Phi = f(\varphi)$, $\Psi = g(\psi)$ for some functions f and g which are assumed to be suitably differentiable. We propose to determine all such flows φ, ψ and Φ, Ψ.

The basic relations between Φ and Ψ, and between φ and ψ are

$$\Phi_x = \Psi_y/\rho, \qquad \Phi_y = \Psi_x/\rho \tag{3a}$$

$$\varphi_x = \psi_y, \qquad \varphi_y = -\psi_x \tag{3b}$$

where $\rho(x, y)$ is the density of the gas at (x, y). It follows from $\Phi = f(\varphi)$, $\Psi = g(\psi)$ that

$$f'(\varphi) = g'(\psi)/\rho \tag{4}$$

From Bernoulli's equation for an ideal gas with polytropic constant γ, we have $\rho = k(1 - f'^2 q^2)^{1/(\gamma-1)}$ q being the velocity (appropriately normalized) of the incompressible flow and k being a fixed reference constant of the gas flow. By including k within the function g', making the substitutions

$$(f')^{\gamma-1} = X(\varphi), \qquad (g')^{\gamma-1} = Y(\psi)$$

and solving for q^2 in (4), we have,

$$q^2 = (X - Y)/X^{(\gamma+1)/(\gamma-1)} \tag{5}$$

Taking the logarithm of both sides, and recalling that $\log q$ is harmonic, and that φ and ψ are conjugate harmonic, we obtain, after then applying the Laplacian operator to both sides and simplifying, that,

$$0 = \frac{(X - Y)(X'' - Y'') - (X'^2 + Y'^2)}{(X - Y)^2} - \left(\frac{\gamma + 1}{\gamma - 1}\right)\frac{XX'' - X'^2}{X^2} \tag{6}$$

where the primes signify differentiation of the functions X, Y with respect to φ, ψ respectively.

Since φ and ψ are independent functions, equation (6) can be considered an ordinary differential equation for Y or X in which the expressions in X or Y, respectively, behave as constants. We wish to determine all functions $Y(\psi)$ and $X(\varphi)$ satisfying (6), where the solutions must be independent of φ, ψ, respectively. Treating (6) as an equation in Y and collecting terms, we have,

$$(Y - X)Y'' - Y'^2 + A_1Y^2 + A_2Y + A_3 = 0, \tag{7}$$

where

$$A_1 = \left(\frac{\gamma + 1}{\gamma - 1}\right)\frac{X'^2 - XX''}{X^2}$$

$$A_2 = \frac{1}{X}\left[\left(\frac{\gamma+3}{\gamma-1}\right)XX'' - 2\left(\frac{\gamma+1}{\gamma-1}\right)X'^2\right]$$

$$A_3 = 2(X'^2 - XX'')/(\gamma - 1).$$

By means of the substitutions $Y'^2 = U$, $Y - X = z$, (7) is reduced to the first order equation in U,

$$(8) \qquad \frac{1}{2}\frac{dU}{dz} - \frac{U}{z} + \frac{A_1X^2 + A_2X + A_3}{z} + 2A_1X + A_2 + A_3z = 0.$$

This is readily integrated to give the general solution,

$$(9) \qquad \tfrac{1}{2}U - \tfrac{1}{2}(A_1X^2 + A_2X + A_3) - z(2A_1X + A_2) + A_1z^2\log z + Cz^2 = 0$$

C being the constant of integration. Let this now be differentiated with respect to ψ, and designate such differentiation with primes; recalling that $z' = -X'$, and that U, Y are independent of ψ, we obtain,

$$(10) \qquad A_1'z^2\log z - 2X'A_1z\log z - z[A_1X' + 2CX' + (2A_1X + A_2)'] + 2A_1XX' + A_2X' - \tfrac{1}{2}(A_1X^2 + A_2X + A_3)' = 0.$$

The above equation is an identity in z which holds for all values of ψ. The following two possibilities are then available:

(A) Y is a constant; equations (6) to (10) are then simply ordinary differential equations for X which can be solved directly.

(B) Y is not constant; in which case the above identity (10) for z, holding for each value of ψ, has as consequence that the coefficients are each zero identically (in ψ) since otherwise $\log z$ could be expressed as a rational function of z for some values of the coefficients. From the vanishing of $X'A_1$ we have either $X =$ constant or $A_1 = 0$. The former will be seen to be the only alternative. For let $A_1 = 0$, with $X' \neq 0$. Then from the vanishing of the other coefficients in (10), and since $A_1 = 0$ implies also $A_3 = 0$, it follows that $A_2 = \bar{C}X$ and $\bar{C} = -2C$ for some constant $\bar{C}$. Combined with $A_1 = 0$, this gives

$$(11) \qquad X' = \sqrt{2C}\,X$$

while equation (9) reduces to

$$(12) \qquad U + 2CY^2 = 0, \text{ or } Y' = \sqrt{-2C}\,Y.$$

Equations (11) and (12) can hold simultaneously for real functions X, Y only if $C = 0$, therefore only if $Y' = X' = 0$, contrary to assumption.

Hence the only solutions to (6) are

$$(A) \qquad Y = \text{constant}, \; X(\varphi) = c_1X^{(\gamma+1)/(\gamma-1)}e^{c_2\varphi} + Y;$$

$$(b) \qquad X = \text{constant}, \; Y(\psi) = c_1e^{c_2\psi} + X.$$

Returning to equation (5), it is seen that these conditions correspond respectively to flows of the type $\psi = \psi(q)$ and $\varphi = \varphi(q)$. Potential flows of the type $\psi = \psi(q)$ were shown in the preceding section to be precisely the circular vortex flows.

In exactly the same way one can show that potential flows of the type $\varphi = \varphi(q)$ are radial source or sink flows. As is well known, the flow patterns of these two types of motion correspond to motions of both compressible and incompressible fluids. According to the above analysis, these are the only such common flow patterns.

The preceding result comparing the flow patterns of potential gas flows with incompressible potential flows can be strengthened to include more general compressible fluids. For example, if, in the equation of state of the fluid, $p = p(\rho)$, the pressure p is an analytic function of the density ρ, one can obtain the same final result with essentially the same methods as those used above on gas flows. This result can undoubtedly be stated even more generally.

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THE COUPLING OF TWO ACOUSTICAL DUCTS

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I. Introduction. A large class of problems which occur in physics and particularly in acoustics[1] require the solution of the scalar Helmholtz equation

$$\nabla^2\psi + \mu^2\psi = 0 \tag{1.1}$$

where ψ is subject to the homogeneous boundary condition

$$\partial\psi/\partial n = f\psi. \tag{1.2}$$

The derivative $\partial\psi/\partial n$ is the normal derivative of ψ on the boundary of the region in which we solve (1.1), while f is a constant. The operator ∇^2 is the three dimensional Laplacian and the time variation which equation (1.1) normally possesses has been suppressed by the assumption that it is simple harmonic, that is $e^{-i\mu ct}$. In acoustics, the f in (1.2) may be complex, corresponding to covering the boundary surface with an acoustical material. Here

$$f = i\omega\rho Y \tag{1.3}$$

where Y is the "admittance" of the material, ρ is the density of the medium enclosed by the boundary surface, $\omega = \mu c$ is the angular frequency of the time variation and c is the velocity of sound in the medium. If Y has a real component, absorption results.

Problems involving the boundary condition (1.2) seem to be considerably more difficult than those involving the more usual boundary conditions: (a) the Dirichlet $\psi = 0$, or (b) the Neumann $\partial\psi/\partial n = 0$. For example, although solutions in closed form can be obtained by the method of separation of variables for the cases (a) and (b) for eleven different coordinate systems, this method will yield solutions for only three coordinate systems, the Cartesian, circular cylindrical, and the spherical for the boundary condition (1.2).

In this paper and several to follow, a number of problems involving mixed boundary conditions will be treated. Here we are still concerned with the solution of equation (1.1) but now the boundary condition (1.2) is modified to read

$$\partial\psi/\partial n = f\psi \tag{1.2a}$$

on a portion of the boundary of the region, while

$$\partial\psi/\partial n = f'\psi \tag{1.2b}$$

on the remainder of the boundary. Such a boundary value problem may be reduced to an integral equation with the aid of an appropriate Green's function. For the special class of geometries which we consider here, this integral equation

[1] See for example P. M. Morse and R. H. Bolt, "Sound Waves in Rooms" Rev. Mod. Phys. 16 (69) 1944.

becomes a Wiener-Hopf[2] integral equation of the second kind. It was first pointed out by J. S. Schwinger[3] that a number of problems in electrodynamics had the appropriate geometries and could be formulated as Wiener-Hopf integral equations of the first kind. The boundary conditions in this group of problems in electrodynamics were either of the Dirichlet or Neumann type. Schwinger together with J. F. Carlson,[4] A. E. Heins[5] and H. Levine have solved numerous problems in the theory of guided waves in this fashion.[6]

The particular problem treated here is the study of the propagation of the principal mode down an infinite duct of rectangular cross section which is free of radiation sources in the finite part of the duct. (See Figure I for a side view). In Figure I, the planes $y = \pm b$ are subject to the boundary conditions.

$$\begin{aligned} \partial\psi/\partial n &= f\psi \qquad x < 0, \quad y = \pm b \\ \partial\psi/\partial n &= f'\psi \qquad x > 0, \quad y = \pm b. \end{aligned} \tag{1.3}$$

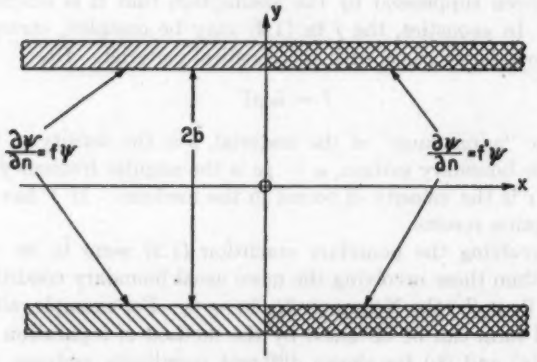

Fig. 1

The two parallel planes $z = \pm a$ will be assumed to be perfectly rigid so that on these latter surfaces $\partial\psi/\partial n = 0$. We are still confronted with specifying the boundary conditions at infinity. To the left of the plane $x = 0$, we assume that we have an incident wave of the form $e^{ik_x x} \cos k_y y \cos \frac{m\pi z}{a}$ exciting this structure, and also a wave reflected from the discontinuity of the form

$$e^{-ik_x x} \cos k_y y \cos \frac{m\pi z}{a}.$$

[2] Paley and Wiener, "Fourier Transforms in the Complex Domain" American Mathematical Society Publications, Vol. XIX, 1934, Chapter IV.

[3] J. S. Schwinger, unpublished.

[4] Carlson and Heins, "The Reflection of an Electromagnetic Plane Wave by an Infinite Set of Plates" I, Quarterly of Applied Mathematics, Vol. IV, P. 313-329, 1947.

[5] Heins and Carlson, "The Reflection of an Electromagnetic Plane Wave by an Infinite Set of Plates" II, forthcoming in the Quarterly of Applied Mathematics.

[6] E. T. Copson, "On an Integral Equation Arising in the Theory of Diffraction." Quarterly Journal of Mathematics, Vol. 17, 1946, P. 19.

Here

$$k^2 = k_x^2 + k_y^2 = \mu^2 - m^2\pi^2/a^2 \ (m \text{ integer})$$

k_y is determined by the transcendental equation

$$(k_y b) \tan k_y b = -fb. \tag{1.4}$$

We assume, henceforth, that kb is sufficiently small so that the only "propagating" mode is the one with the smallest magnitude of $k_y b$. For example, if the walls $y = \pm b$ are rigid for $x < 0$, so that $f = 0$, then $k_y = 0$ and the incident wave will assume a functional form independent of y, $e^{ik_x z} \cos \frac{m\pi z}{a}$. If f is complex, the imaginary part of k_x must be less than the imaginary part of the higher modes but greater than zero. For x large and positive, ψ assumes the form of a wave travelling in the positive x direction, i.e.

$$e^{ik'_x z} \cos k'_y y \cos \frac{m\pi z}{a}$$

where now

$$k_x'^2 + k_y'^2 = \mu^2 - m^2\pi^2/a^2 \qquad (m \text{ integer})$$

and k'_y is determined from a transcendental equation of the form (1.4),

$$k'_y b \tan k'_y b = -bf'. \tag{1.5}$$

We shall assume that f and f' are so related that $Imk_x < Imk'_x$. For small values of Imk_x and Imk'_x, this would require $Imf < Imf'$. This is an unimportant limitation, for the treatment of the opposite case differs only in minor detail from the present discussion.

Because of the symmetry of the principal mode with respect to y about the plane $y = 0$, and also the symmetry of the duct with respect to the same plane, it is clear that $\partial\psi/\partial n = 0$ for $y = 0$. The problem we have described above is then equivalent to the coupling of two semi-infinite ducts, the lower surface of which is rigid, while the respective upper surfaces are of different acoustical material. Thus Figure II gives a side view of the geometric structure for the reduced problem.

The boundary conditions at infinity and at $z = \pm a$ remain unaltered.

We note, finally, that it is possible to remove the z dependence from the equation (1.1) since we know the correct functional form of the variation. If we write

$$\psi(x, y, z) = \varphi(x, y) \cos \frac{m\pi z}{a}$$

we are left with

$$\partial^2\varphi/\partial x^2 + \partial^2\varphi/\partial y^2 + k^2\varphi = 0 \tag{1.6}$$

which is to be solved subject to the boundary conditions

$$\partial\varphi/\partial y = f\varphi \qquad y = b,\ x < 0 \tag{1.7}$$

$$\partial\varphi/\partial y = f'\varphi \qquad y = b,\ x > 0 \tag{1.8}$$

$$\partial\varphi/\partial y = 0 \qquad y = 0,\ -\infty < x < \infty \tag{1.9}$$

plus the corresponding conditions for x large which we have described for the function $\psi(x, y, z)$.

2. Reduction of the Boundary Value Problem to a Wiener-Hopf Integral Equation. In order to reduce equation (1.6) and the boundary conditions (1.7), (1.8) and (1.9) to an integral equation of the Wiener-Hopf type, we apply Green's Theorem to a region in the xy plane bounded by the rectangle $y = 0$, $y = b$, $x = c_1$ and $x = c_2$ $(c_1 < c_2)$, which will enable us to express $\varphi(x, y)$ inside the rectangle in terms of a Green's function appropriate to the rectangle, the function φ and its normal derivative on the boundary. Thus

$$\varphi(x, y) = \oint \left[G(x, y; x', y') \frac{\partial\varphi}{\partial n'} - \varphi \frac{\partial G(x, y; x', y')}{\partial n'} \right] ds' \tag{2.0}$$

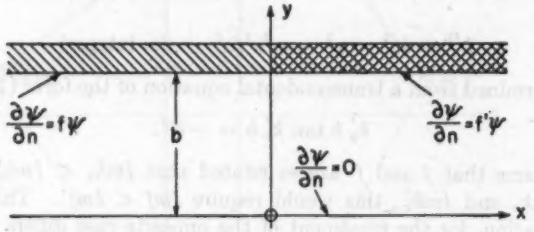

Fig. 2

where the φ under the integral sign is a function of the coordinates x' and y'. The integral in (2.0) is to be treated as an improper line integral. That is, we first consider the closed path to be a rectangle whose boundaries are $y = b$, $y = 0$, $x = c_1$ and $x = c_2$. After having evaluated the integral along this path, we then allow $c_1 \to -\infty$ and $c_2 \to \infty$.

$G(x, y; x', y')$ satisfies the partial differential equation

$$\partial^2 G/\partial x^2 + \partial^2 G/\partial y^2 + k^2 G = 0 \tag{2.1}$$

save for the point $x = x'$, $y = y'$. At this point the two first derivatives of G suffer discontinuities such that

$$\int_{-\infty}^{\infty} \frac{\partial G}{\partial y'} \Big|_{y'-y-0}^{y'-y+0} dx' = -1 \tag{2.2a}$$

$$\int_{0}^{b} \frac{\partial G}{\partial x'} \Big|_{x'-x-0}^{x'-x+0} dy' = -1\,. \tag{2.2b}$$

We may express the conditions (2.2a) and (2.2b) symbolically by saying that $G(x, y, x', y')$ satisfies the inhomogeneous partial differential equation

$$\partial^2 G/\partial x^2 + \partial^2 G/\partial y^2 + k^2 G = -\delta(x - x')\delta(y - y') \tag{2.3}$$

where $\delta(x - x')$ and $\delta(y - y')$ are the Dirac delta functions. The boundary conditions which we impose on $G(x, y; x', y')$ at the surfaces $y = 0$ and $y = b$ are dictated, in part, by the simple fact that we should like to make (2.0) as simple an integral equation as possible. To this end, we note that if we choose $\partial G/\partial y' = 0$ at $y' = 0$, we eliminate immediately the integral along the path $y' = 0, -\infty < x' < \infty$.

On the surface $y' = b$, we use the boundary condition $\partial G/\partial y' = f G$. The choice of this boundary condition on the Green's function for the surface $y' = b$ eliminates the integral along the path $y' = b, x' < 0$. In order to calculate the contributions which arise from the surfaces $x' = c_1$ and c_2, we find it convenient to introduce the Green's function explicitly, though actually only the asymptotic form is needed in the ensuing discussion:

$$G(x, y; x', y') = \sum_{n=1}^{\infty} \frac{e^{-\mu_n|x-x'|} \cos \sigma_n y \cos \sigma_n y'}{2\mu_n N_n} + \frac{\sin k_x |x - x'| - \sin k_x(x - x')}{k_x N_0} \cos k_y y' \cos k_y y$$

where

$$\mu_n = \sqrt{\sigma_n^2 - k^2}, \qquad N_n = \int_0^b \cos^2 \sigma_n y \, dy$$

and the σ_n are the roots of the equation (1.4) with positive real parts. The term $\sin k_x(x - x')$ in the Green's function is, of course, a solution of the homogeneous wave equation, which, as we shall see, it is particularly convenient to introduce as we do.

We note that for $x \gg x'$, $G(x, y; x', y')$ is asymptotic to

$$[e^{-\mu_1(x-x')} \cos \sigma_1 y \cos \sigma_1 y']/2\mu_1 N_1$$

while for $x \ll x'$, it is asymptotic to

$$(2/k_x N_0) \sin k_x(x' - x) \cos k_y y' \cos k_y y.$$

It is to be further noted that for $x \ll 0$, $\varphi(x, y)$ is asymptotic to

$$(\alpha_1 e^{ik_x x} + \beta_1 e^{-ik_x x}) \cos k_y y$$

while for $x \gg 0$, $\varphi(x, y)$ is asymptotic to

$$\alpha_2 e^{ik_x' x} \cos k_y' y.$$

Hence (2.0) reduces to

$$\varphi(x, y) = \int_0^{c_2} \left[G(x, y; x', y') \frac{\partial \varphi}{\partial y'} - \varphi(x', y') \frac{\partial G(x', y'; x, y)}{\partial y'} \right]_{y'=b} dx' + \frac{2\alpha_2 \cos k_y y}{k_x N_0} \int_0^b \cos k_y y' \cos k_y' y' \cdot [ik_x' \sin k_x(x - x') - k_x \cos k_x(x' - x)]_{x'=c_2} e^{ik_x' x'} \, dy' .$$

In virtue of our assumption that $Imk_x < Imk_x'$, the second term in (2.4) approaches zero as c_2 becomes infinite so that we are finally left with

$$\varphi(x, y) = (f' - f) \int_0^\infty G(x, y; x', b')\varphi(x', b)\, dx' . \tag{2.5}$$

We obtain the desired integral equation by evaluating (2.5) at $y = b$, that is

$$\varphi(x, b) = (f' - f) \int_0^\infty G(x, b; x', b)\varphi(x', b)\, dx'. \tag{2.6}$$

(2.6) is a Wiener-Hopf integral equation because of its limits and the $x - x'$ variation of its kernel, $G(x, b; x', b)$.

3. Definition of the Reflection Coefficient. On the wall $y = b$, the incident wave has the amplitude $\alpha_1 \cos k_y b$, while the reflected wave has the amplitude $\beta_1 \cos k_y b$. We now propose to evaluate α_1 and β_1 in terms of an integral over $\varphi(x', b)$, to be more precise the Fourier transform of $\varphi(x', b)$. Since we solve equation (2.6) by Fourier transform methods, these quantities will appear directly in our solution. We shall define the ratio β_1/α_1 as the reflection coefficient, that is, the ratio of the amplitude of the reflected to the incident wave.

In the first place, since $\varphi(x, b)$ is asymptotic to $(\alpha_1 e^{ik_x x} + \beta_1 e^{-ik_x x}) \cos k_y b$ for $x \ll 0$, we see from (2.6) that

$$\cos k_y b(\alpha_1 e^{ik_x x} + \beta_1 e^{-ik_x x}) = 2\, \frac{f' - f}{k_x N_0} \cos^2 k_y b \int_0^\infty \sin k_x(x' - x)\varphi(x', b)\, dx' . \tag{3.0}$$

It therefore follows from (3.0) that

$$\alpha_1 = i\, \frac{(f' - f)}{k_x N_0} \cos k_y b \int_0^\infty e^{-ik_x x'}\varphi(x', b)\, dx'$$

$$\beta_1 = -i\, \frac{(f' - f)}{k_x N_0} \cos k_y b \int_0^\infty e^{ik_x x'}\varphi(x', b)\, dx' .$$

Hence, the reflection coefficient is

$$r = \beta_1/\alpha_1 = -\int_0^\infty e^{ik_x x'}\varphi(x', b)\, dx' \Big/ \int_0^\infty e^{-ik_x x'}\varphi(x', b)\, dx', \tag{3.1}$$

or if $\bar{\varphi}(k)$ is the unilateral Fourier transform of $\varphi(x, b)$,

$$r = -\bar{\varphi}(-k_x)/\bar{\varphi}(k_x). \tag{3.2}$$

4. The Fourier Transform Solution of Equation (2.6). The application of the Wiener-Hopf theory to equation (2.6) requires a redefinition of the function $\varphi(x, b)$, and an investigation of the regions of regularity of the Fourier transforms of $\varphi(x, b)$ and $G(x, b; x', b)$. Because of the structure of equation (2.6) in which the integral involves the values of $\varphi(x, b)$ for $x > 0$ only, it is possible to consider

the behavior of $\varphi(x, b)$ for positive values of x and negative values of x separately. Let us retain the symbol $\varphi(x, b)$ for the values of $\varphi(x, b)$ for $x > 0$ so that

$$\varphi(x, b) \equiv 0$$

for $x < 0$. For negative values of x we replace φ by a function $P(x)$ which naturally satisfies $P(x) \equiv 0$ for $x > 0$. In terms of these functions, the integral equation (2.6) may now be written

$$P(x) + \varphi(x, b) = (f' - f) \int_0^\infty G(x, b; x', b)\varphi(x', b)\, dx'. \tag{4.1}$$

We shall now investigate the growth orders of $P(x)$, $\varphi(x, b)$ and $G(x, b; 0, b)$ as $x \to \infty$. We have already remarked that the function $\varphi(x, b)$ is asymptotic to $e^{ik_x'x}$ ($Imk_x' > 0$) for x large and positive. The function $P(x)$ for x large and negative is asymptotic to $e^{ik_x x}$ as one may see from (4.1). Finally $G(x, b; x', b)$ is asymptotic to $e^{-\mu_1(x-x')}$ $x \gg x'$ and to $e^{ik_x(x-x')}$ for $x \ll x'$. Noting the growth order of these functions at infinity and the added information that these functions $\varphi(x, b)$ and $P(x)$ are integrable for any finite x, we are in a position to determine the regions of regularities of the various Fourier transforms which are required. For example, since $\varphi(x, b) \equiv 0$ for $x < 0$, its Fourier transform

$$\int_0^\infty e^{-i\omega x}\varphi(x, b) = \bar{\varphi}(\omega)$$

is regular in the half plane $Im\omega < Imk_x'$. Similarly the transform of $P(x)$,

$$\int_0^\infty e^{-i\omega x}P(x)\, dx = \bar{P}(\omega)$$

is regular in the half plane $Im\omega > Imk_x$. Finally, the transform of $G(x, b; 0, b)$,

$$\int_0^\infty e^{-i\omega x}G(x, b; 0, b)\, dx = \bar{G}(\omega, b, b)$$

is regular in the strip $Imk_x < Im\omega < Re\mu_1$. Thus, the Fourier transforms of $P(x)$, $\varphi(x, b)$ and $G(x, b, 0, b)$ are regular in the strip $Imk_x < Im\omega < Imk_x'$ or $Re\mu_1$ depending which of the quantities Imk_x' or $Re\mu_1$ is smaller. (Actually the physical assumptions were such that $Imk_x' < Re\mu_1$.) It is thus legitimate to apply the Fourier transform theorem to the equation (4.1). This gives immediately

$$\bar{\varphi}(\omega) = \frac{f' \cos \sigma b + \sigma \sin \sigma b}{f \cos \sigma b + \sigma \sin \sigma b} = -\bar{P}(\omega) \tag{4.2}$$

where now $\sigma = \sqrt{k^2 - \omega^2}$.
This employs the result

$$\bar{G}(\omega, b, b) = -\frac{\cos \sigma b}{\sigma \sin \sigma b + f \cos \sigma b}.$$

The Wiener-Hopf theory now tells us that we are required to decompose

$$(f' \cos \sigma b + \sigma \sin \sigma b)/(f \cos \sigma b + \sigma \sin \sigma b) \tag{4.3}$$

multiplicatively into two functions which are regular[7] in appropriate half planes. Both the numerator and denominator of (4.3) have an infinite sequence of zeros. Denote those in the denominator by σ_n and those in the numerator by σ_n'. These roots[8] are the roots of equations (1.4) and (1.5) respectively. Equation (4.3) may be rewritten in product form as

$$\left\{f' \prod_{n=0}^{\infty} [1 - (\sigma/\sigma_n')^2]\right\} \Big/ \left\{f \prod_{n=0}^{\infty} [1 - (\sigma/\sigma_n)^2]\right\}.$$

For n sufficiently large, the roots $b\sigma_n$ and $b\sigma_n'$ are asymptotic to $n\pi - bf/n\pi$ and $n\pi - bf'/n\pi$ respectively. Now we rewrite (4.3) so that its zeros and poles are put into evidence, that is

$$\frac{f' \prod_{n=0}^{\infty}\left[1 - \frac{k^2 - \omega^2}{\sigma_n'^2}\right]}{f \prod_{n=0}^{\infty}\left[1 - \frac{k^2 - \omega^2}{\sigma_n^2}\right]} = \frac{\sigma_0^2}{(\sigma_0')^2} \frac{f'}{f} \frac{(\omega - k_x')(\omega + k_x')}{(\omega - k_x)(\omega + k_x)} \cdot \frac{\prod_{n=1}^{\infty} [\sqrt{1 - k^2/\sigma_n'^2} + i\omega/\sigma_n'] e^{-i\omega b/n\pi} \prod_{n=1}^{\infty}\left[\sqrt{1 - k^2/\sigma_n'^2} - \frac{i\omega}{\sigma_n'}\right] e^{i\omega b/n\pi}}{\prod_{n=1}^{\infty} [\sqrt{1 - k^2/\sigma_n^2} + i\omega/\sigma_n] e^{-i\omega b/n\pi} \prod_{n=1}^{\infty}\left[\sqrt{1 - k^2/\sigma_n^2} - \frac{i\omega}{\sigma_n}\right] e^{i\omega b/n\pi}}.$$

The factors $e^{\pm i\omega b/n\pi}$ have been inserted in the above products to insure their absolute convergence. The choice of these factors is not unique since we may choose any exponent which is asymptotic to $\pm i\omega b/n\pi$ as n becomes infinite. The final answer is, of course, independent of this particular choice.

As a matter of notation let us write

$$\Pi(\sigma', \omega) = \prod_{n=1}^{\infty} [\sqrt{1 - k^2/(\sigma_n')^2} + i\omega/\sigma_n'] e^{-i\omega b/n\pi}. \tag{4.4}$$

Then, in this abbreviated notation (4.3) becomes

$$\frac{\sigma_0^2}{(\sigma_0')^2} \frac{f'}{f'} \frac{(\omega - k_x')(\omega + k_x')}{(\omega - k_x)(\omega + k_x)} \frac{\Pi(\sigma', \omega)\Pi(\sigma', -\omega)}{\Pi(\sigma, \omega)\Pi(\sigma, -\omega)}.$$

The terms $\Pi(\sigma, \omega)$ and $\Pi(\sigma', \omega)$ are regular in the lower half plane $Im\omega < Imi\mu^1$, while the terms $\Pi(\sigma, -\omega)$ and $\Pi(\sigma', -\omega)$ are regular in the upper half plane $Im\omega > -Imi\mu_1$. Furthermore, the terms $(\omega + k_x')$, $(\omega - k_x)$ and $\omega + k_x$ are regular in the upper half plane $Im\omega > Imk_x$, while the term $\omega - k_x'$ is regular in the lower half plane $Im\omega < Imk_x'$.

Upon noting the above decomposition of (4.2) into factors which are regular

[7] We use the term "regular" in a slightly extended sense. We imply that both the function and its reciprocal are devoid of singularities in its region of regularity.

[8] These roots have been tabulated. See Morse, Lowan, Feshbach and Haurwitz, "Tables for Solutions of the Wave Equation for Rectangular and Circular Boundaries Having Finite Impedance."

in certain upper and lower half planes, we are now in a position to effect the final decomposition of equation (4.2). We write

$$\bar{P}(\omega)\frac{(\omega-k_x)(\omega+k_x)}{\omega+k_x'}\frac{\Pi(\sigma,-\omega)}{\Pi(\sigma',-\omega)}=-\bar{\varphi}\frac{\sigma_0^2}{(\sigma_0')^2}\frac{f'}{f}(\omega-k_x')\frac{\Pi(\sigma',\omega)}{\Pi(\sigma,\omega)}. \tag{4.5}$$

The left side of equation (4.5) is regular in the upper half plane $Im\omega > Imk_x$, while the right side is regular in the lower half plane $Im\omega < Imk_x'$ (or $Re\mu_1$). Both sides are regular in the strip $Imk_x < Im\omega < Imk_x'$ (or $Re\mu_1$) and are therefore analytical continuations of each other. Such a state of regularity can only exist if both sides of equation (4.5) are equal to an integral function. That is, denoting this integral function by $E(\omega)$ we have

$$E(\omega)=-\frac{\bar{P}(\omega)(\omega-k_x)(\omega+k_x)\Pi(\sigma,-\omega)}{(\omega+k_x')\Pi(\sigma',-\omega)} \tag{4.6a}$$

$$E(\omega)=\frac{\bar{\varphi}(\omega)\sigma_0^2 f'}{\sigma_0'^2 f}(\omega-k_x')\frac{\Pi(\sigma',\omega)}{\Pi(\sigma,\omega)} \tag{4.6b}$$

In order to determine $E(\omega)$, we find it necessary to find the asymptotic forms of the II functions, $\bar{\varphi}(\omega)$ and $\bar{P}(\omega)$ for $|\omega|\to\infty$ in the appropriate half plane of regularity. Let us consider the function $\Pi(\sigma,\omega)$. For $|\omega|\to\infty$, $Im\omega < Im i\mu_1$ we can show that this product is dominated by the reciprocal of the gamma function. To show this, we note that if we choose $n = N$ such that $kb/N\omega \ll 1$ and $fb/(N\pi)^2 \ll 1$, the product

$$\Pi(\sigma,\omega)=\prod_{n=1}^{\infty}[\sqrt{1-k^2/\sigma_n^2}+i\omega/\sigma_n]e^{-i\omega b/n\pi}$$

may be approximated by

$$\Pi(\sigma,\omega)\simeq\prod_{n=1}^{N}\left[\frac{\sqrt{1-k^2/\sigma_n^2}+i\omega/\sigma_n}{1-\dfrac{k^2b^2}{2n^2\pi^2}+\dfrac{i\omega b}{n\pi}\left(1+\dfrac{fb}{n^2\pi^2}\right)}\right]\prod_{n=1}^{\infty}\left[1-\frac{k^2b^2}{n^2\pi^2}+\frac{i\omega b}{n\pi}\left(1+\frac{fb}{n^2\pi^2}\right)\right]$$

since the major contribution in the infinite product comes from the term $i\omega b/n\pi$ in the parenthesis. The infinite product may be rewritten in factor form so that

$$\Pi(\sigma,\omega)\simeq\prod_{n=1}^{N}\left[\frac{\sqrt{1-k^2/\sigma_n^2}+i\omega/\sigma_n}{1-\dfrac{k^2}{n^2\pi^2}+\dfrac{i\omega b}{n\pi}\left(1+\dfrac{fb}{n^2\pi^2}\right)}\right]\prod_{n=1}^{\infty}(1-x_1/n\pi)\,e^{x_1/n\pi}\cdot$$

$$\cdot\prod_{n=1}^{\infty}(1-x_2/n\pi)\,e^{x_2/n\pi}\prod_{n=1}^{\infty}(1-x_3/n\pi)e^{x_3/n\pi}$$

where now x_1, x_2, and x_3 are roots of the equation

$$x^3+x^2\,i\omega b-\tfrac{1}{2}k^2b^2x+i\omega b^2 f=0\,. \tag{4.8}$$

For $|\omega|$ large, it is quite easy to show that the roots of (4.7) are asymptotic to

$$x_1=-i\omega b+2i\alpha/\omega b$$

$$x_2 = i\sqrt{fb} - i\alpha/\omega b \qquad \alpha = \tfrac{1}{2}fb + \tfrac{1}{4}k^2b^2 . \tag{4.9}$$
$$x_3 = -i\sqrt{fb} - i\alpha/\omega b$$

Upon noting the functional form of the gamma function, (4.7) may be rewritten as

$$\frac{\delta e^{-i\omega b\gamma/n\pi}}{\Gamma(1 - x_1/n\pi)\Gamma(1 - x_2/\pi)\Gamma(1 - x_3/\pi)} \tag{4.10}$$

where γ is the Euler-Mascheroni constant; δ is of the form of $A + B/\omega$ where A and B are constants depending only on f and b. It is clear then, that for $|\omega| \to \infty$, $Im\omega < Imi\mu_1$, the ratio

$$\Pi(\sigma, \omega)/\Pi(\sigma', \omega) \tag{4.11}$$

is constant since the terms in $1/\omega$ and the constant terms (4.10) are secondary in importance to those in ω.

We now return to equation (4.6b) and note that $\varphi(x, b)$ is integrable at the origin. That is, $\varphi(x, b) = Ax^\alpha$, $\alpha > -1$, $x \to 0^+$. The Fourier transform of $\varphi(x, b)$ then behaves like $\omega^{-\alpha-1}$ for $|\omega| \to \infty$, $Im\omega < Imi\mu_1$. Hence (4.6b) is asymptotic to $E(\omega) \to \rho\omega^{-\alpha}$. Since $E(\omega)$ is an entire function of algebraic growth, α is therefore a non-negative integer. If we repeat this argument with equation (4.6a) we shall again find that $E(\omega)$ is asymptotic to $\omega^{-\alpha}$ for $|\omega| \to \infty$, $Im\omega < Imi\mu_1$. If α is a positive integer, $E(\omega)$ will vanish identically and therefore $\bar{P}(\omega)$ and $\bar{\varphi}(\omega)$ will do the same. We are then forced to conclude that $\alpha = 0$ and hence $E(\omega)$ is a constant. The constant is immaterial since any solution of the integral equation may be multiplied by a constant and still remain a solution.

We are now in a position to find the function $\varphi(x, b)$ from its known Fourier transform $\bar{\varphi}$. However, we shall content ourselves with a discussion of $\varphi(x, b)$ as $x \to 0^+$ since the quantity of most interest, the reflection coefficient, may be simply expressed in terms of $\bar{\varphi}$ according to (3.2).

The behavior of $\varphi(x, b)$ as $x \to 0^+$ depends upon the behavior of $\bar{\varphi}$ as $|\omega| \to \infty$. We therefore expand $\bar{\varphi}$ in inverse powers of ω. To do this, we merely need apply the Stirling expansion formula to (4.11) and make use of (4.10). This gives immediately that $\Pi(\sigma, \omega)/\Pi(\sigma', \omega) \simeq \rho(1 + \theta(\log\omega)/\omega)$ where ρ and θ are parameters depending on fb, $f'b$ and kb. Therefore

$$\bar{\varphi}(\omega) \sim \frac{1}{\omega}\left(1 - \frac{\theta \log \omega}{\omega}\right)$$

and hence

$$\varphi(x, b) \sim 1 - x(\theta_1 + \theta_2 \log x) \ (x \to 0^+).$$

It is clear then, that for $x \to 0$, $\varphi(x, b)$ is finite, but its x derivative has a logarithmic singularity. The function $P(x)$ which we introduced into eq. (4.1) is actually the continuation of $\varphi(x, b)$ for $x < 0$. The argument we have employed above will demonstrate that $\varphi(x, b) \to P(x)$ as $x \to 0$ and will further show that $\varphi(x, b)$ and $P(x)$ will have derivatives which are infinite in a logarithmic fashion and are of opposite sign.

5. Reflection Coefficient. The most useful result, from a physical point of view, which may be obtained from the above analysis is the reflection coefficient r. This may be obtained directly from (3.2) and (4.6) since the integral function $E(\omega)$ is a constant. One finds:

$$r = \frac{k_x - k_x'}{k_x + k_x'} \frac{\prod(\sigma, -k_x)}{\prod(\sigma, k_x)} \frac{\prod(\sigma', k_x)}{\prod(\sigma', -k_x)}. \tag{5.1}$$

The first factor $(k_x - k_x')/(k_x + k_x')$ is precisely the reflection coefficient one would compute in the "transmission line" approximation[9] in which the effects of the higher modes $\mu_n (n > 1)$ are neglected. These modes must enter into the exact solution in order that the proper match be made between the modes in that part of the tube corresponding to $x < 0$ and those in the region $x > 0$. If these modes are neglected a good approximation may be obtained only if the discontinuity in the boundary condition $(f - f')\, b \ll 1$, for then the strength of these higher modes is correspondingly small.

The actual calculation of the reflection coefficient can be carried out numerically and will be published elsewhere[10]. We shall content ourselves here with the derivation of some approximate formulas but the methods employed apply directly to the cases where only numerical methods will suffice.

We may rewrite the reflection coefficient as follows:

$$r = \frac{k_x - k_x'}{k_x + k_x'} \frac{\prod(\sigma, -k_x)\prod(\sigma, k_x)}{\prod(\sigma', -k_x)\prod(\sigma', k_x)} \frac{\prod^2(\sigma', k_x)}{\prod^2(\sigma, k_x)}.$$

Consider the general product $\Pi\,(\sigma, -\omega)\ \Pi\,(\sigma, \omega)$. This equals

$$\prod_{n=1}^{\infty}\left[1 - \frac{k^2 - \omega^2}{\sigma_n^2}\right] = \frac{\sigma_0^2(f \cos \sigma b + \sigma \sin \sigma b)}{f(\sigma_0^2 - \sigma^2)}.$$

Thus

$$\begin{aligned}\prod(\sigma', -k_x)\prod(\sigma', k_x) &= \frac{(\sigma_0')^2}{f'}\left[\frac{f' \cos \sigma_0 b + \sigma_0 \sin \sigma_0 b}{(\sigma_0')^2 - \sigma_0^2}\right] \\ &= \frac{(\sigma_0')^2 \cos \sigma_0 b}{f'}\left[\frac{f' - f}{(\sigma_0')^2 - \sigma_0^2}\right].\end{aligned} \tag{5.2}$$

To evaluate $\Pi\,(\sigma, -k_x)\ \Pi\,(\sigma, k_x)$ we use a simple limiting process, obtaining:

$$\begin{aligned}\prod(\sigma, -k_x)\prod(\sigma, k_x) &= \lim_{\sigma \to \sigma_0} \frac{\sigma_0^2}{f} \frac{f \cos \sigma b + \sigma \sin \sigma b}{\sigma_0^2 - \sigma^2} \\ &= \frac{\cos \sigma_0 b}{2f}\,[f - bf^2 - \sigma_0^2 b].\end{aligned} \tag{5.3}$$

[9] See Feshbach and Harris, "The Effect of Non-Uniform Wall Distributions of Absorbing Materials on the Acoustics of Rooms." J. A. S. A. 18, 472 (1946).

[10] To be submitted to the Journal of the Acoustical Society of America.

Finally

$$\frac{\Pi(\sigma, -k_z)\Pi(\sigma, k_z)}{\Pi(\sigma', -k_z)\Pi(\sigma', k_z)} = \left(\frac{f'}{2f}\right)\frac{(f - bf^2 - \sigma_0^2 b)[(\sigma_0')^2 - \sigma_0^2]}{(\sigma_0')^2(f' - f)}. \tag{5.4}$$

To continue further, analytically, it becomes necessary, particularly for the evaluation of the Π^2 term, to introduce the following approximations:

$$|bf| << 1, |bf'| << 1, kb << 1. \tag{5.5}$$

In terms of these, $(\sigma_0 b)^2 \simeq -[fb + \frac{1}{3}(fb)^2 + \frac{4}{45}(fb)^3]$ and

$$\frac{\Pi(\sigma, -k_z)\Pi(\sigma, k_z)}{\Pi(\sigma, ' -k_z)\Pi(\sigma_, ' k_z)} \simeq 1 + \frac{1}{90}[(fb)^2 - (f'b)^2] + \frac{1}{90}[fb - f'b]^2.$$

The next terms are third or higher order in bf and (bf')

Consider now, $\Pi(\sigma, k_z)$ under approximations (5.5.):

$$\Pi(\sigma, k_z) = \exp\left\{\sum_{n=1}^{\infty} \ln\left[\sqrt{1 - \frac{k^2}{\sigma_n^2}} + \frac{ik_z}{\sigma_n}\right] - \frac{ik_z b}{n\pi}\right\}.$$

Expanding the square root, replacing k^2 by $k_z^2 + (\sigma_0 b)^2$ we obtain neglecting third and higher powers of fb:

$$\Pi(\sigma, k_z) \simeq \exp\left\{\sum_{n=1}^{\infty} \ln\left[1 + \frac{ik_z b}{n\pi} - \frac{1}{2}\frac{(k_z b)^2}{n^2\pi^2} + \frac{fb}{2n^2\pi^2} + (fb)^2\left(\frac{1}{6n^2\pi^2} + \frac{7}{8n^4\pi^4}\right) + \frac{i(k_z b)(fb)}{n^3\pi^3} - \frac{3}{4}\frac{(k_z b)^2 fb}{n^4\pi^4}\right] - \frac{ik_z b}{n\pi}\right\}.$$

Upon expansion of the logarithm and squaring we find

$$\Pi^2(\sigma, k_z) \simeq \exp\left\{\sum_{n=1}^{\infty} 2\ln\left(1 + \frac{ik_z b}{n\pi} - \frac{1}{2}\frac{(k_z b)^2}{(n\pi)^2}\right) + \frac{fb}{n^2\pi^2} + (fb)^2\left[\frac{1}{3}\frac{1}{n^2\pi^2} + \frac{7}{4n^4\pi^4}\right] + \frac{i(k_z b)(fb)}{n^3\pi^3} - \frac{3}{2}\frac{(k_z b)^2(fb)^2}{n^4\pi^4} - \frac{2ik_z b}{n\pi}\right\}.$$

Finally

$$\frac{\Pi^2(\sigma', k_z)}{\Pi^2(\sigma, k_z)} \simeq 1 + (f' - f)b\sum_1^{\infty}\frac{1}{n^2\pi^2} + [(f'b)^2 - (fb)^2]\left[\frac{1}{3}\sum_1^{\infty}\frac{1}{n^2\pi^2} + \frac{7}{4}\sum_1^{\infty}\frac{1}{n^4\pi^4}\right] + \tfrac{1}{2}(f' - f)^2 b^2\left(\sum_1^{\infty}\frac{1}{n^2\pi^2}\right)^2 + i(k_z b)(f' - f)b\sum_1^{\infty}\frac{1}{n^3\pi^3} - \tfrac{1}{2}(k_z b)^2(f' - f)b\sum_1^{\infty}\frac{1}{n^4\pi^4}. \tag{5.7}$$

Introducing (5.6) and (5.7) into the expression for the reflection coefficient r and again omitting terms of third order or greater, one obtains

$$r \simeq \frac{k_z - k_z'}{k_z + k_z'}\left\{1 + \frac{(f' - f)b}{6} + [(f'b)^2 - (fb)^2]\left(\frac{23}{360}\right) + \frac{(f' - f)^2b^2}{40}\right. \tag{5.8}$$
$$\left. + \frac{i\zeta(3)(k_z b)(f' - f)b}{\pi^2} - \frac{1}{60}(k_z b)^2(f' - f)b\right\}$$

where ζ (3) is the Riemann Zeta function of 3 and has the numerical value 1.202. The deviation from the transmission line value is, as we surmised earlier, proportional to $(f' - f)b$.

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THE ACCURACY OF THE ROOT-SQUARING METHOD FOR SOLVING EQUATIONS

By P. G. Hoel and D. D. Wall

1. Introduction. In a recent paper,[1] Bodewig advocated the root-squaring method for solving polynomial equations largely on the basis of its ability to yield both real and imaginary roots and its property of giving all the roots with very little more work than is required to obtain one root.

A feature of the method that was brought to our attention by Mr. Bell of Lockheed Aircraft and which should make it of even greater value is the possibility of performing the bulk of the computations with punched-card equipment. In solving the equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0, \tag{1}$$

the root-squaring method proceeds by forming a new equation

$$x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n = 0, \tag{2}$$

in which the roots are the negatives of the squares of the roots of (1). The coefficients of (2) are related to those of (1) by the formula

$$b_k = a_k^2 - 2a_{k-1}a_{k+1} + 2a_{k-2}a_{k+2} - \cdots \tag{3}$$

Since the operations in (3) can be performed with punched-card equipment, the process of obtaining an equation whose roots are the negatives of a high power of the original roots can be reduced to a large extent to purely mechanical operations. When this powered equation is available, the computations necessary to obtain the roots of (1) become quite brief. They are described by Bodewig.

Because of the diverse errors that can arise in the root-squaring method, it was considered important by individuals employing the method to study the magnitude of such errors. Although Bodewig considered the approximate rate of error decrease in a certain situation, he did not give any serious consideration to the problems of accuracy. This paper is concerned with such problems.

There are two important sources of error in the root-squaring method. One arises from not powering equation (1) to a sufficiently high power. The other arises from a loss of coefficient accuracy in the powered equation because of carrying out the operations in (3) to a limited accuracy. The latter error is the rounding error that is introduced by the limited digital capacity of calculating equipment. In this paper, bounds will be obtained for these two kinds of errors.

2. Power errors. Let

$$x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_n = 0 \tag{4}$$

[1] E. Bodewig, *On Graeffe's method of solving algebraic equations*, Quarterly Appl. Math. **4**, 177–190 (1946).

denote the powered equation after s squarings and let $m = 2^s$. If the roots of (1) are ordered with respect to the magnitudes of their absolute values so that $|r_{i+1}| \leq |r_i|$, then

$$c_k = \sum r_1^m r_2^m \cdots r_k^m,$$

where the summation extends over all combinations of products of roots taken k at a time and only the leading term is displayed. To further simplify the notation, let $t_i = r_i^m$, then

$$c_k = \sum t_1 t_2 \cdots t_k . \tag{5}$$

2.1 *Real Distinct Roots.* In this section, bounds for r_k will be obtained in terms of the c's for equations with real roots only.

Consider the ratio

$$\frac{c_k c_{k-2}}{c_{k-1}^2} = \frac{\sum t_1 \cdots t_k \sum t_1 \cdots t_{k-2}}{[\sum t_1 \cdots t_{k-1}]^2}.$$

If the largest term in each summation is factored out, it will follow that

$$\frac{c_k c_{k-2}}{c_{k-1}^2} = \frac{t_k}{t_{k-1}} \frac{[1 + t_{k+1}/t_k + \cdots][1 + t_{k-1}/t_{k-2} + \cdots]}{[1 + t_k/t_{k-1} + \cdots]^2}$$

$$\geq \frac{t_k}{t_{k-1}} \left[1 + \frac{t_k}{t_{k-1}} \left\{\binom{n}{k-1} - 1\right\}\right]^{-2}.$$

Consequently,

$$\frac{t_k}{t_{k-1}} \leq \frac{c_k c_{k-2}}{c_{k-1}^2} \left[1 + \frac{t_k}{t_{k-1}} \left\{\binom{n}{k-1} - 1\right\}\right]^2.$$

Let $\alpha = c_k c_{k-2}/c_{k-1}^2$ and $\beta = \binom{n}{k-1} - 1$; then if α is sufficiently small, this inequality will be equivalent to the inequality

$$\frac{t_k}{t_{k-1}} \leq \frac{1 - 2\alpha\beta - (1 - 4\alpha\beta)^{\frac{1}{2}}}{2\alpha^2\beta}.$$

If the radical is expanded, it will be found that this inequality implies the following inequality.

$$\frac{t_k}{t_{k-1}} \leq \alpha \frac{1 - 2\alpha\beta}{1 - 4\alpha\beta}. \tag{6}$$

By analogy, if $\alpha' = c_{k+1}c_{k-1}/c_k^2$ and $\beta' = \binom{n}{k} - 1$,

$$\frac{t_{k+1}}{t_k} \leq \alpha' \frac{1 - 2\alpha'\beta'}{1 - 4\alpha'\beta'}. \tag{7}$$

Next, consider the ratio

$$\frac{c_k}{c_{k-1}} = \frac{\sum t_1 \cdots t_k}{\sum t_1 \cdots t_{k-1}} = t_k \frac{[1 + t_{k+1}/t_k + \cdots]}{[1 + t_k/t_{k-1} + \cdots]}.$$

In the same manner as before,

$$t_k[1 + \beta t_k/t_{k-1}]^{-1} \leq \frac{c_k}{c_{k-1}} \leq t_k[1 + \beta' t_{k+1}/t_k],$$

which is equivalent to

$$\frac{c_k}{c_{k-1}}[1 + \beta' t_{k+1}/t_k]^{-1} \leq t_k \leq \frac{c_k}{c_{k-1}}[1 + \beta t_k/t_{k-1}].$$

By means of (6) and (7), these inequalities yield

$$\frac{c_k}{c_{k-1}}\left[1 + \alpha'\beta' \frac{1 - 2\alpha'\beta'}{1 - 4\alpha'\beta'}\right]^{-1} \leq t_k \leq \frac{c_k}{c_{k-1}}\left[1 + \alpha\beta \frac{1 - 2\alpha\beta}{1 - 4\alpha\beta}\right].$$

Since $t_k = r_k^m$, the desired bounds for r_k become

$$\left(\frac{c_k}{c_{k-1}}\right)^{1/m}\left[1 + \alpha'\beta' \frac{1 - 2\alpha'\beta'}{1 - 4\alpha'\beta'}\right]^{-(1/m)} \leq |r_k| \leq \left(\frac{c_k}{c_{k-1}}\right)^{1/m}\left[1 + \alpha\beta \frac{1 - 2\alpha\beta}{1 - 4\alpha\beta}\right]^{1/m}. \tag{8}$$

These bounds involve only quantities that are available when (4) is available. Since the root-squaring method employs $(c_k/c_{k-1})^{1/m}$ as its estimate of $|r_k|$, the maximum relative error of this estimate due to an insufficient number of squarings can be obtained easily by means of (8). These bounds also hold for the distinct roots of (1) even though there may be equal roots present; however, they are not applicable to the equal roots.

2.2 *Real Distinct Roots With Bounded Ratios.* The bounds given by (8) do not enable the computer to determine in advance how many squarings will be necessary to obtain a desired accuracy. Such bounds will be derived in this section.

It will be assumed here that there exists a number $r < 1$ such that for $i = 1, 2, \cdots, n - 1$, $|r_{i+1}| \leq r|r_i|$. Then $|r_j| \leq r^{j-i}|r_i|$ for $j > i$. If $t = r^m$, it therefore follows from the definition of t_i that for $j > i$

$$t_j \leq t^{j-i} t_i. \tag{9}$$

Let $t_{s_1} t_{s_2} \cdots t_{s_k}$ denote a typical term of (5). Then, by means of (9), it follows that

$$\frac{t_{s_1} t_{s_2} \cdots t_{s_k}}{t_1 t_2 \cdots t_k} \leq t^{(s_1-1)+(s_2-2)+\cdots+(s_k-k)} \leq t^{s_1+\cdots+s_k-\frac{1}{2}k(k+1)},$$

and therefore that

$$c_k \leq t_1 t_2 \cdots t_k \sum t^{s_1+\cdots+s_k-\frac{1}{2}k(k+1)}, \tag{10}$$

where the summation extends over all positive integers $s_1 < s_2 < \cdots < s_k \leq n$.

The number of terms in (5) that will give rise to t^p in (10) is therefore equal to the number of solutions of the equation

$$s_1 + \cdots + s_k = p + \tfrac{1}{2}k(k+1), \tag{11}$$

in which the s_i are positive integers satisfying $s_1 < s_2 < \cdots < s_k \leq n$. The number of such solutions can be found by means of generating functions. This problem will be considered after an expression for the bounds for the roots has been obtained.

Let the polynomial in t in (10) be denoted by $1 + Q(t)$. Since the coefficients of $Q(t)$ are positive integers,

$$t_1 t_2 \cdots t_k \leq c_k \leq t_1 t_2 \cdots t_k[1 + Q(t)]. \tag{12}$$

If $Q'(t)$ corresponds to $Q(t)$ for c_{k-1}, these inequalities will yield

$$t_k[1 + Q'(t)]^{-1} \leq c_k/c_{k-1} \leq t_k[1 + Q(t)].$$

Consequently,

$$(c_k/c_{k-1})[1 + Q(t)]^{-1} \leq t_k \leq (c_k/c_{k-1})[1 + Q'(t)],$$

or expressed in terms of r_k,

$$\left(\frac{c_k}{c_{k-1}}\right)^{1/m}[1 + Q(t)]^{-(1/m)} \leq |r_k| \leq \left(\frac{c_k}{c_{k-1}}\right)^{1/m}[1 + Q'(t)]^{1/m}. \tag{13}$$

Although these bounds involve coefficients of (4), the relative error terms depend only on t, k, n, and m; consequently if the value of t, or at least an upper bound for it, is available, these bounds can be used to determine the minimum number of squarings to guarantee a given accuracy in r_k.

Now consider the postponed problem of evaluating $Q(t)$, which in turn depends upon determining the number of solutions of (11). It is known[2] that the number of such solutions is equal to the coefficient of y^p in the polynomial generating function

$$P_k(y) = \frac{(1 - y^{n-k+1})(1 - y^{n-k+2}) \cdots (1 - y^n)}{(1 - y)(1 - y^2) \cdots (1 - y^k)}. \tag{14}$$

Thus, in any given problem it is possible to determine $Q(t)$ explicitly. Since it would be laborious to determine $Q(t)$ for each value of k, it will be shown that $Q(t)$ will be a maximum when $k = n/2$ if k is even, and when $k = (n-1)/2$ if k is odd. This maximum $Q(t)$ will then yield bounds for all the roots.

Consider the generating function $P_{k+1}(y)$ when k is increased by one. It would be obtained by multiplying $P_k(y)$ by

$$(1 - y^{n-k})/(1 - y^{k+1}). \tag{15}$$

Since the only zeros of $P_k(y)$ are roots of unity, not including unity itself, and since such roots must arise from factors of the form

$$1 + y^{pi} + y^{2pi} + \cdots + y^{mipi},$$

[2] P. A. MacMahon, *Combinatory analysis*, vol. 2, Cambridge University Press, 1916, p.10.

it follows that when $P_k(y)$ is reduced to lowest terms, it may be expressed in the form

$$P_k(y) = \prod_i (1 + y^{p_i} + y^{2p_i} + \cdots + y^{m_i p_i}). \tag{16}$$

Similarly, if (15) is reduced to lowest terms, it may be expressed in the form

$$\frac{1 - y^{n-k}}{1 - y^{k+1}} = \frac{1 + y^q + y^{2q} + \cdots + y^{\lambda q}}{1 + y^q + y^{2q} + \cdots + y^{\mu q}}. \tag{17}$$

In case $n - k$ is a multiple of $k + 1$, $\mu = 0$. Since $P_{k+1}(y)$ is given as the product of (16) and (17) and is a polynomial in y, the denominator of (17) must occur as a factor in (16). Therefore $P_k(y)$ can be written as

$$P_k(y) = (1 + y^q + y^{2p} + \cdots + y^{\mu q}) \prod_i{}' (1 + y^{p_i} + y^{2p_i} + \cdots + y^{m_i p_i}), \tag{18}$$

where the indicated product here differs from the indicated product in (16) by this one factor. Now $P_{k+1}(y)$ is the product of (17) and (18); therefore

$$P_{k+1}(y) = (1 + y^q + y^{2q} + \cdots + y^{\lambda q}) \prod_i{}' (1 + y^{p_i} + y^{2p_i} + \cdots + y^{m_i p_i}).$$

Since, from (17), $\lambda > \mu$ for $k < (n - 1)/2$, this shows that $P_{k+1}(y)$ will contain all the terms of $P_k(y)$ and additional terms. Thus, the coefficients in $P_k(y)$ will be monotone increasing functions of k for $k < (n - 1)/2$. Therefore, using symmetry, $P_k(y)$ will possess maximum coefficients when $k = n/2$ if n is even, and when $k = (n - 1)/2$ if n is odd. Because of this result, it is necessary to determine $Q(t)$ in (13) only for this value of k in order to obtain bounds that are valid for all the roots.

As an illustration of how (13) is applied, suppose that $n = 10$, $r = .9$, and four squarings are performed. Then $m = 2^4 = 16$ and $t = (.9)^{16} = .01853$. From considerations in the preceding paragraph, k will be chosen as 5. From (14),

$$1 + Q(t) = \frac{(1 - t^6)(1 - t^7) \cdots (1 - t^{10})}{(1 - t)(1 - t^2) \cdots (1 - t^5)}.$$

If this is expressed in explicit polynomial form, it will be found that

$$1 + Q(t) = 1 + t + 2t^2 + 3t^3 + 5t^4 + 7t^5 + 9t^6 + \cdots + t^{25}. \tag{19}$$

Calculations will verify that $1 + Q(t) \leq 1.01923$ and that $[1 + Q(t)]^{1/16} \leq 1.0012$; hence (13) becomes

$$(c_k/c_{k-1})^{1/16}[1.0012]^{-1} \leq |r_k| \leq (c_k/c_{k-1})^{1/16}[1.0012].$$

Thus, the relative error after four squarings will not exceed .0012.

2.3 *Real And Imaginary Roots With Bounded Ratios.* In this section imaginary roots will also be permitted; however, it will be assumed that the distribution of the absolute values of the roots is such that (9) will hold for

roots with different absolute values. For example, if r_i is real and r_{i+1} and r_{i+2} represent a pair of complex conjugate roots, it would be necessary that $|r_{i+2}| \leq r^2 |r_i|$. The analysis of the preceding section then requires only slight modifications to become valid. The right side of (12) will be true if absolute value signs are inserted around the t_i and c_k. If $Q(t) < 1$, the left side of (12) may be replaced by

$$|c_k| \geq |t_1||t_2| \cdots |t_k|[1 - Q(t)],$$

and therefore (12) may be replaced by

$$|t_1||t_2| \cdots |t_k|[1 - Q(t)] \leq |c_k| \leq |t_1||t_2| \cdots |t_k|[1 + Q(t)].$$

The analogue of (13) will therefore become

$$(20) \qquad \left|\frac{c_k}{c_{k-1}}\right|^{1/m}\left[\frac{1+Q(t)}{1-Q(t)}\right]^{-1/m} \leq |r_k| \leq \left|\frac{c_k}{c_{k-1}}\right|^{1/m}\left[\frac{1+Q(t)}{1-Q(t)}\right]^{1/m}.$$

It is unnecessary to use a prime on $Q(t)$ in the right inequality, as was done in (13), if it is understood that the maximum $Q(t)$ is chosen.

2.4 *Rate Of Error Decrease.* Inequalities (20), and for only real roots (13), may be used to estimate the rate of convergence of the root-squaring method. If one additional squaring is performed, (20) will be replaced by

$$\left|\frac{c_k'}{c_{k-1}'}\right|^{1/(2m)}\left[\frac{1+Q(t^2)}{1-Q(t^2)}\right]^{-(1/2m)} \leq |r_k| \leq \left|\frac{c_k'}{c_{k-1}'}\right|^{1/(2m)}\left[\frac{1+Q(t^2)}{1-Q(t^2)}\right]^{1/(2m)},$$

where c_k' corresponds to c_k. If a sufficient number of squarings have been performed to make t so small that its higher powers may be ignored, then

$$\left[\frac{1+Q(t)}{1-Q(t)}\right]^{1/m} = \left[\frac{1+ct}{1-ct}\right]^{1/m} \approx 1 + \frac{2ct}{m},$$

whereas

$$\left[\frac{1+Q(t^2)}{1-Q(t^2)}\right]^{1/2m)} = \left[\frac{1+ct^2}{1-ct^2}\right]^{1/(2m)} \approx 1 + \frac{ct^2}{m},$$

where c is the coefficient of t in $Q(t)$. This shows that the relative error will be decreased by the factor $t/2$ if one more squaring is performed. Contrary to Bodewig's claim, the number of exact decimals is not doubled with every squaring because the factor $1/m$ is very small for situations in which the preceding approximations are valid. For early squarings, the rate of decrease can be estimated in any given problem by evaluating $Q(t)$ and $Q(t^2)$ to the desired accuracy.

Inequalities (20) are also convenient for determining the number of squarings necessary to obtain any desired accuracy, provided a satisfactory estimate of r is available. For example, if $r = .9$ and a relative error not to exceed .0001 is desired in an equation of the tenth degree or lower which contains both real and imaginary roots, then m would need to satisfy

$$\left[\frac{1+Q(t)}{1-Q(t)}\right]^{1/m} - 1 \leq .0001.$$

From (19), this inequality is equivalent to

$$\left[\frac{1 + (.9)^m + 2(.9)^m + \cdots}{1 - (.9)^m - 2(.9)^m - \ldots}\right]^{1/m} - 1 \leq .0001.$$

Calculations will show that six squarings will suffice to satisfy the inequality.

3. Rounding errors. The results in the preceding sections are based on the assumption that the operations involved are carried out without computational errors; however, because of the limited carrying capacity of calculating equipment, errors of rounding may be introduced. In this section, bounds will be obtained for such errors.

3.1 *Real Distinct Roots With Bounded Ratios.* The assumptions here will be the same as those in 2.2.

Consider the magnitude of c_k' relative to that of c_k^2, where c_k' corresponds to c_k after one more squaring. Since $c_k' = \sum l_1^2 l_2^2 \cdots l_k^2$, it follows from (5) that

$$c_k' \leq c_k^2. \tag{21}$$

The coefficients, c_i, will usually possess the same digital accuracies, at least in the early squarings, if the a_i possess the same digital accuracies. As a consequence, the product terms in

$$c_k' = c_k^2 - 2c_{k-1}c_{k+1} + 2c_{k-2}c_{k+2} - \cdots$$

would be expected to possess the same digital accuracy as c_k^2; however, they will ordinarily possess some different decimal accuracies.

If the product terms possess at least as much decimal accuracy as c_k^2, then c_k' would be expected to possess the same decimal accuracy as c_k^2, unless n is very large. The product terms will possess this property provided that

$$c_k^2 \geq 2c_{k-j}c_{k+j} \tag{22}$$

for $1 \leq j \leq k$, because they will possess the same digital accuracy as c_k^2 but will not exceed c_k^2 in magnitude. Evidence will be presented in the second following paragraph to show that (22) will usually be satisfied.

Now decimal accuracy will correspond to digital accuracy provided that c_k' and c_k^2 are of the same order of magnitude. There will be a loss in digital accuracy if c_k' is much smaller than c_k^2. The possibility of losing digital accuracy can therefore be determined by studying the ratio c_k^2/c_k'. If this ratio is approximately equal to 10^h, then a loss of approximately h digits in accuracy would be expected.

Before proceeding to a study of this ratio, consider the reasonableness of assuming that (22) will be satisfied. If the roots are widely separated, as after several squarings, then

$$\frac{c_k^2}{2c_{k-j}\,c_{k+j}} \cong \frac{l_{k-j+1} \cdots l_k}{l_{k+1} \cdots l_{k+j}}, \tag{23}$$

which is very large because the numerator factors dominate the corresponding

denominator factors. At the other extreme, if the roots are all equal, then

$$\frac{c_k^2}{2c_{k-j}c_{k+j}} = \frac{\binom{n}{k}^2}{2\binom{n}{k-j}\binom{n}{k+j}}.$$

It can be shown that this ratio will possess a minimum when $j = 1$ and $k = n/2$ if n is even, and $k = (n-1)/2$ if n is odd. If these minimizing values are employed, it will be found that (22) will be satisfied for $n > 4$.

Now consider the ratio c_k^2/c_k'. Since

$$\frac{c_k^2}{c_k'} = \frac{[\sum t_1 t_2 \cdots t_k]^2}{\sum t_1^2 t_2^2 \cdots t_k^2} = \frac{[1 + t_{k+1}/t_k + \cdots]^2}{[1 + t_{k+1}^2/t_k^2 + \cdots]}, \tag{24}$$

it follows that

$$c_k^2/c_k' \leq [1 + t_{k+1}/t_k + \cdots]^2,$$

and hence that

$$c_k^2/c_k' \leq [1 + Q(t)]^2. \tag{25}$$

As an illustration, the example considered in 2.2 will give

$$c_k^2/c_k' \leq 1.039;$$

consequently at this late stage in the squaring process no accuracy will be lost beyond that normally to be expected in multiplying and adding numbers of a fixed digital accuracy.

3.2 *Equal Roots.* If all the roots are equal, (24) will become

$$c_k^2/c_k' = \binom{n}{k}^2 \Big/ \binom{n}{k} = \binom{n}{k}.$$

Here there will be a rapid loss of digital accuracy. Thus, for $n = 10$ and $k = 5$, $c_k^2/c_k' = 252$; consequently two digits would be expected to be lost in one squaring operation. Since this property holds for each operation, a few such operations would give rise to an equation with no accuracy remaining in its coefficients. From this result it is clear that there is danger of losing accuracy as roots approach equality.

3.3 *Real And Imaginary Roots With Bounded Ratios.* Inequality (25) can be modified in the same manner that (13) was modified to yield (20). In addition to the assumptions made in 2.3, it is also necessary to assume that (21) and (22) are satisfied for absolute values because the arguments leading to (25) depend upon these inequalities. A loss of accuracy because (21) is not satisfied seems highly improbable. As indicated in 3.2, the dangerous situations are those in which (21) is very much satisfied. Although (22) may not be satisfied in the early squarings, arguments similar to those following (23) will show that (22) may be expected to hold in the later squarings. If these conditions

are satisfied, the analogue of (25) will be

$$\frac{c_k^2}{|c_k'|} \leq \frac{[1 + Q(t)]^2}{[1 - Q(t^2)]}.$$

4. Other errors. A third source of error that has not been considered in this paper is the rounding error of routine multiplications and additions of approximate numbers. If the number of digits carried in the calculations is several more than the number of digits in the original coefficients, this error need not give the computer concern, unless n is very large. Although the original coefficients may be in error, equation (1) is treated here as an exact equation and therefore such extra digits are justified.

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NUMERICAL DETERMINATION BY USE OF SPECIAL COMPUTATIONAL DEVICES OF AN INTEGRAL OPERATOR IN THE THEORY OF COMPRESSIBLE FLUIDS*

II. Determination of the Coefficients of the Integral Operator by Interpolatory Means

By Rufus Isaacs

1. Introduction. In this paper there is presented a method for computing by use of a special digiting device (such as punch card machines, an automatic sequence controlled calculator, Eniac, etc.) the coefficients $Q^{(n)}$ of an integral operator, introduced by Bergman, in the theory of compressible fluids. Use of this operator will make possible the calculation of a two-dimensional subsonic flow pattern of a compressible fluid. The underlying theory is outlined in Section 1 of the first paper in this series.[1]

In Section 2 the functions are displayed and reduced to a form where they have a range and domain suitable for computational purposes. The functions are defined by a recurrence relation involving derivatives and integrals. Using a procedure suggested by Professor von Mises, they are to be computed by interpolatory means, i.e., they are to be replaced by expressions involving nothing more than a finite number of arithmetical operations. Such a replacement involves an error. Bounds for this error can be given in terms of the higher order derivatives of the functions. In Section 3 bounds for the required derivatives are obtained.

In Section 4, the interpolatory formulas themselves are derived and their coefficients and bounds for the error computed. In Section 5, the interpolation formulas are applied to the functions. The computational formulas and error bounds are given in a final form, ready for use.

This paper includes the following specific results:

1. The formulas to be used on the calculator.
2. Tables of coefficients for the use of the above.
3. Precise bounds for the arising error given in both tabular and formula form.

These results are summarized in Section 6.

The author wishes to extend his thanks to Professor von Mises for his valuable help and advice in carrying out the present work.

2. The Function Sequence. The functions whose values are ultimately sought are denoted by $Q^{(n)}(\lambda)$, $(n = 0, 1, 2, \cdots ; -\infty < \lambda \leqq 0)$ and they are

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[1] See this Journal, vol. 25 pp. 1–9.

defined by[2] $Q^{(0)}(\lambda) = 1$

$$(2n+1)Q^{(n+1)}(\lambda) = -(d/d\lambda)Q^{(n)}(\lambda) - 4\int_{-\infty}^{\lambda} F(\alpha)Q^{(n)}(\alpha)\,d\alpha, \qquad (n = 0, 1, 2, \cdots) \tag{2.1}$$

As will subsequently appear, we work not with the functions $Q^{(n)}$, but with others simply related to them whose range and domain are more conveniently bounded.

In (2.1)

$$F(\lambda) = \frac{1-T^2}{16(1-h^2)^2T^6}[5 - (1+6h^2)T^2 - (5-4h^2)T^4 + (1+2h^2)T^6] = \frac{1-T^2}{16(1-h^2)^2T^6}\zeta(T) \tag{2.2}$$

and T is related to λ by

$$\lambda = \tfrac{1}{2}\log\left[\frac{1-T}{1+T}\left(\frac{1+hT}{1-hT}\right)^{\frac{1}{h}}\right] \tag{2.3}$$

and h is a gas constant approximately equal to 6^{-1}.

Instead of λ we may use as independent variable $x = 1 - T$ with the range $[0, 1]$. We introduce first the functions

$$s_n(x) = (-1)^n(1-h^2)^n(1\cdot 3\cdot 5\cdot \cdots \cdot(2n-1))Q^{(n)}(\lambda).$$

Then it is easy to verify

$$s_0(x) = 1,\ s_{n+1}(x) = \mathfrak{A}(x)s'_n(x) + \int_0^x \mathfrak{B}(\alpha)s_n(\alpha)d\alpha \tag{2.4}$$

where

$$\mathfrak{A}(x) = \frac{1}{T^2} + h^2T^2 - (1+h^2), \qquad \mathfrak{B}(x) = \frac{1}{4T^4(1-h^2T^2)}\zeta(T) \tag{2.5}$$

The function s_n becomes infinite as T nears zero. In fact it can be shown that there exist constants $c_0, c_1, \cdots$ such that

$$\lim_{T\to 0} T^{3n}s_n = c_n. \tag{2.6}$$

We shall remove this difficulty by first ascertaining the c_n and then setting $r_n(T) = T^{3n}s_n/c_n$ so that $r_n(0) = 1$. The $r_n(T)$ are the functions that will actually be calculated.

Determination of the c_n. (2.6) implies

$$s_n = (c_n/T^{3n})[1 + \phi_n(T)] \tag{2.7}$$

[2] See Bergman, §11, p. 35, of [2]; §2 and Appendix I of [5]. The numbers in brackets refer to the Bibliography.

See also formulas (1.4) and (2.1) of the first article in this series.

with $\phi_n(0) = 0$. Substituting (2.7) into (2.4) we obtain

$$\frac{c_{n+1}}{c_n}(1 + \phi_{n+1}(T)) = 3nT^2\mathfrak{A}(T)(1 + \phi_n(T)) - T^3\mathfrak{A}(T)\phi_n'(T) + T^{3n+3}\int_T^1 \frac{\mathfrak{B}(\alpha)(1 + \phi_n(\alpha))}{\alpha^{3n}}\,d\alpha \qquad \{T\}^3. \tag{2.8}$$

Now $\mathfrak{B}(T) = \frac{1}{T^4}f(T)$ where $f(T)$ is bounded in $[0, 1]$ and $f[0] = \frac{5}{4}$. Put $q_n(T) = f(T)(1 + \phi_n(T))$ so that the integral in (2.8) is

$$\int_T^1 \frac{q_n(\alpha)\,d\alpha}{\alpha^{3n+4}} = -\frac{1}{3n+3}\frac{q_n(\alpha)}{\alpha^{3n+3}}\Big]_T^1 + \frac{1}{3n+3}\int_T^1 \frac{q_n'(\alpha)}{\alpha^{3n+3}}\,d\alpha$$

$$= \frac{1}{3n+3}\left[\frac{q_n(T)}{T^{3n+3}} - q_n(1)\right] - \frac{3n+2}{3n+3}q_n(\theta)\left[\frac{1}{T^{3n+2}} - 1\right], \qquad (T \leqq \theta \leqq 1).$$

Then

$$\operatorname*{Lim}_{T\to 0} T^{3n+3}\int_T^1 \frac{q_n(\alpha)}{\alpha^{3n+4}}\,d\alpha = \frac{q_n(0)}{3n+3} = \frac{5}{12(n+1)}.$$

It follows that if we let $T \to 0$ in (2.8)

$$\frac{c_{n+1}}{c_n} = \left(3n + \frac{5}{12(n+1)}\right) = \frac{1}{u_n} \tag{2.9}$$

so that

$$c_0 = 1, \qquad c_1 = \tfrac{5}{12}, \qquad c_n = \tfrac{5}{12}\left(3 + \frac{5}{12\cdot 2}\right)\cdot\left(3\cdot 2 + \frac{5}{12\cdot 3}\right)\cdots\left(3(n-1) + \frac{5}{12n}\right) \quad \text{for } n > 1.$$

Finally we obtain, by substitution in (2.4) the recurrence relations for the r_n:

$$r_0(T) = 1,$$

$$r_{n+1}(T) = u_n\left[3nA(T)r_n(T) - TA(T)r_n'(T) + T^{3n+3}\int_T^1 \frac{B(\alpha)}{\alpha^{3n}}r_n(\alpha)\,d\alpha\right] \qquad \{T\} \tag{2.10}$$

where

$$A(T) = 1 - (1 + h^2)T^2 + h^2T^4, \qquad B(T) = \mathfrak{B}(T). \tag{2.11}$$

It is to the equations (2.10) that we shall apply the interpolation procedure.

[3] When a variable appears in braces to the right of a relation, it means that derivatives and integrals appearing in or implied by this relation are with respect to this variable. The $\mathfrak{A}(T)$ and $\mathfrak{B}(T)$ in (2.8) mean the right side of (2.5).

3. Bounds for the Derivatives. The interpolation formulas that we thall use in the computations of the r_n contain error terms which involve high order derivatives of the r_n. Thus it is necessary to obtain bounds for $|r_n^{(k)}(T)|$.[4]

By a *dominant* of a function $f(x)$ (for the interval $0 \leqq x \leqq 1$) we shall mean a function $\bar{f}(x)$ such that

$$|f^{(k)}(x)| \leqq \bar{f}^{(k)}(x) \quad (k = 0, 1, 2, \cdots; 0 \leqq x \leqq 1) \tag{3.1}$$

and write

$$f << \bar{f} \tag{3.2}$$

THEOREM 1.[5] *If*

$$\mathfrak{A} << \bar{\mathfrak{A}}, \qquad \mathfrak{B} << \bar{\mathfrak{B}} \quad \{x\} \tag{3.3}$$

and $h_n (n = 0, 1, 2, \cdots)$ is a non-negative constant and the functions $s_n(x)$ are defined by

$$\mathfrak{s}_0(x) = 1, \tag{3.4}$$

$$\mathfrak{s}_{n+1}(x) = \bar{\mathfrak{A}}(x)\mathfrak{s}_n'(x) + \int_0^x \bar{\mathfrak{B}}(\alpha)\mathfrak{s}_n(\alpha)\,d\alpha + h_n \qquad \{x\} \tag{3.5}$$

then

$$s_n << \mathfrak{s}_n \ (n = 0, 1, 2, \cdots) \qquad \{x\} \tag{3.6}$$

PROOF by induction on n. For $n = 0$: — trivial. Assuming $s_n << \mathfrak{s}_n$,

$$|s_{n+1}| = \left|\mathfrak{A}s_n' + \int_0^x \mathfrak{B}s_n\,d\alpha\right| \leqq \bar{\mathfrak{A}}\mathfrak{s}_n' + \int_0^x \bar{\mathfrak{B}}\mathfrak{s}_n\,d\alpha + h_n = \mathfrak{s}_{n+1}$$

while if $k > 0$

$$|s_{n+1}^{(k)}| = \left|\sum_j \binom{k}{j} \mathfrak{A}^{(j)} s_n^{(k-j)} + \sum_j \binom{k-1}{j} \mathfrak{B}^{(j)} s_n^{(k-1-j)}\right|$$

$$\leqq \sum_j \binom{k}{j} \bar{\mathfrak{A}}^{(j)} \mathfrak{s}_n^{(k-j)} + \sum_j \binom{k-1}{j} \bar{\mathfrak{B}}^{(j)} \mathfrak{s}_n^{(k-1-j)} = \mathfrak{s}_{n+1}^{(k)}.$$

THEOREM 2. *Let $p(T) = \sum_{n=0}^{\infty} a_n T^n$ be a power series converging for $|T| \leqq 2$ and let $p^*(T)$ be a majorant, i.e.*

$$p^*(T) = \sum_{n=0}^{\infty} b_n T^n \qquad \text{with} \qquad |a_n| \leqq b_n, \tag{3.7}$$

also convergent for $|T| \leqq 2$. Then

$$\frac{p(T)}{T^m} \ll \frac{p^*(2)}{T^m}. \qquad \{x\} \tag{3.8}$$

[4] Superscripts in parentheses (except in the case of the $Q^{(n)}$) denote derivatives.

[5] See Bergman [2], pp. 35–38.

Let $f(T)$ be such that for some constant $c = c(m)$

$$|f^{(k)}(T)| \leqq \frac{c(k+m-1)!}{T^{k+m}}$$

$$(k = 0, 1, 2, \cdots ; m = 1, 2, \cdots ; 0 \leqq T \leqq 1) \qquad \{x \text{ or } T\}.$$

We show first

$$|(f(T)p(T))^{(k)}| \leqq p^*(2T)\frac{c(k+m-1)!}{T^{k+m}}.$$

$$|(f(T)T^n)^{(k)}| = \left|\sum_j \binom{k}{j} f^{(k-j)}(T^n)^{(j)}\right|$$

$$\leqq c\sum_j \binom{k}{j}\frac{(k-j+m-1)!}{T^{k+m-j}}\frac{n!}{(n-j)!}T^{n-j} = \frac{ck!\,T^n}{T^{k+m}}\sum_j \frac{n!(k-j+m-1)!}{j!(k-j)!(n-j)!}$$

$$= \frac{ck!\,T^n}{T^{k+m}}\sum_j \binom{n}{j}(k-j+m-1)\cdots(k-j+1)$$

$$< \frac{ck!\,T^n(k+m-1)\cdots(k+1)}{T^{k+m}}\sum_j \binom{n}{j}$$

$$= \frac{c(k+m-1)!}{T^{k+m}}(2T)^n$$

$$|(f(T)p(T))^{(k)}| = \left|\sum a_n(f(T)T^n)^{(k)}\right| \leqq \sum b_n |(f(T)T^n)^{(k)}|$$

$$\leqq \frac{c(k+m-1)!}{T^{k+m}}\sum b_n(2T)^n = \frac{c(k+m-1)!}{T^{k+m}}p^*(2T).$$

As $\left(\frac{1}{T^m}\right)^{(k)} = \frac{(k+m-1)!}{(m-1)!T^{k+m}}$, $\qquad \{x\}$

our result follows:

$$\left|\left(\frac{1}{T^m}p(T)\right)^{(k)}\right| \leqq \frac{(k+m-1)!}{(m-1)!T^{k+m}}p^*(2T) \leqq \frac{(k+m-1)!}{(m-1)!\,T^{k+m}}p^*(2)$$

$$= \left(\frac{p^*(2)}{T^m}\right)^{(k)}. \qquad \{x\} \tag{3.9}$$

Theorem 2 is applicable to $\mathfrak{A}$ and $\mathfrak{B}$ and leads to results of the form

$$\mathfrak{A} << C_1/T^2 = \tilde{\mathfrak{A}}, \qquad \mathfrak{B} << C_2/T^4 = \tilde{\mathfrak{B}} \qquad \{x\}. \tag{3.10}$$

Then by an application of Theorem 1 with suitable h_n, simple dominants can be found for s_n. These lead in turn to bounds for the derivatives of r_n. But first we shall find values of C_1 and C_2.

Determination of C_1 and C_2. Because $\mathfrak{A}$ is quite a simple function it is possible to obtain the best possible value of C_1 by directly computing the ratios of the successive derivatives of $\mathfrak{A}$ and T^{-2} and then using the maximum absolute value which appears. This leads to

$$C_1 = 1 + (h^2/3). \tag{3.11}$$

From (2.2) and (2.5) we obtain easily

$$4T^4\mathfrak{B} = 5 - (1+h^2)T^2 + (-5+3h^2-h^4)T^4 + \frac{(1-h^2)^3T^6}{1-h^2T^2}. \tag{3.12}$$

Thus in this case

$$4p^*(2) = 89 - 44h^2 + 16h^4 + \frac{64(1-h^2)^3}{1-4h^2} \cong 195 < 196. \tag{3.13}$$

or we may take

$$C_2 = 49. \tag{3.14}$$

Now we define the following constants

$$\left.\begin{aligned} a &= 3C_1, \qquad b = \tfrac{1}{4}C_2 \\ \pi_0 = 1,\ \pi_1 &= b_1,\ \pi_n \\ &= b\left(a+\frac{b}{2}\right)\left(2a+\frac{b}{3}\right)\cdots\left((n-1)a+\frac{b}{n}\right)\ (n=2,3,\cdots) \end{aligned}\right\} \tag{3.15}$$

If we choose

$$h_n = \frac{b}{n+1}\,\pi_n, \tag{3.16}$$

then suitable δ_n will be given by

$$\delta_n(x) = \pi_n/T^{3n} \tag{3.17}$$

Proof by induction on n.

$\delta_0 = 1$ by definition.

$$\begin{aligned} \delta_{n+1} &= \frac{a}{3T^2}\delta_n' + 3b\int_0^x \frac{\delta_n(\alpha)}{(1-\alpha)^4}\,d\alpha + \frac{b}{n+1}\pi_n \qquad \{x\} \\ &= \frac{a}{3T^2}\frac{3n\pi_n}{T^{3n+1}} + \frac{3b}{3n+3}\pi_n\left(\frac{1}{T^{3n+3}}-1\right) + \frac{b}{n+1}\pi_n \qquad (3.18) \\ &= \frac{\pi_n}{T^{3n+3}}\left(an+\frac{b}{n+1}\right) = \frac{\pi_{n+1}}{T^{3(n+1)}}. \end{aligned}$$

We can now obtain the sought bound for $|r_n^{(k)}|$.

$$r_n(T) = T^{3n}\delta_n/c_n, \tag{3.19}$$

$$\left.\begin{aligned} |r_n^{(k)}(T)| &= \frac{1}{c_n}\left|\sum_j \binom{k}{j}\delta_n^{(j)}\,(T^{3n})^{(k-j)}\right| \qquad \{T\} \\ &\le \frac{\pi_n}{c_n}\sum_j \binom{k}{j}\frac{(3n+j-1)!}{(3n-1)!}\,\frac{1}{T^{3n+j}}\,\frac{(3n)!}{(3n+j-k)!}\,T^{3n-k+j} \\ &= \alpha_{nk}/T^k \end{aligned}\right\} \tag{3.20}$$

where

$$\alpha_{nk} = 3n\,\pi_n/c_n \sum_j \binom{k}{j} [(3n+j-1)\cdots(3n+j-(k-1))] \tag{3.21}$$

Let us consider the quantity in the bracket. As we may suppose $n > 0$, the first factor is positive. As the factors are consecutive integers, there cannot be a negative one present without there also being a zero. Thus the bracket $\geqq 0$ and clearly then $\leqq (3n+k-1)\cdots(3n+1)$. Therefore

$$\alpha_{nk} \leqq \frac{\pi_n}{c_n} 2^k \frac{(3n+k-1)!}{(3n-1)!} \tag{3.22}$$

In addition to our result

$$|\, r_n^{(k)}(T)\,| \leqq \frac{\pi_n}{c_n}\left(\frac{2}{T}\right)^k \frac{(3n+k-1)!}{(3n-1)!} \tag{3.23}$$

we shall later need another bound, namely one for

$$\sigma_{m,n,k} = \sum_{j=m}^{M-1} \left(\frac{B(\xi_j) r_n(\xi_j)}{\xi_j^{3n}}\right)^{(2k)}$$

with $(j-k+1)\omega \leqq \xi_j \leqq (k+j)\omega$.

As will appear later, m and M are positive integers with $M-1 > m > k$ and $\omega > 0$.

We note

$$\frac{B(T)r_n(T)}{T^{3n}} = \frac{B(T)s_n(T)}{c_n}.$$

and

$$\begin{aligned} |\,(B(T)s_n(T))^{(k)}\,| &\leqq 3b\pi_n \sum_j \binom{k}{j} (-1)^j \left(\frac{1}{T^4}\right)^{(j)} (-1)^{k-j}\left(\frac{1}{T^{3n}}\right)^{(k-j)} \quad \{T\} \\ &= (-1)^k 3b\pi_n \left(\frac{1}{T^{3n+4}}\right)^{(k)} = 3b\pi_n \frac{(3n+3+k)!}{(3n+3)!}\frac{1}{T^{3n+4+k}} \end{aligned} \tag{3.24}$$

The smallest admissible value of ξ_j is $\omega\,(j-k+1)$; therefore

$$|\,\sigma_{m,n,k}\,| \leqq \frac{3b\pi_n}{c_n}\frac{(3n+3+2k)!}{(3n+3)!}\sum_{j=m}^{M-1}\frac{1}{[(j-k+1)\omega]^{3n+4+k}}$$

The sum in this expression cannot exceed

$$\begin{aligned} \frac{1}{\omega}\int_{(m-1)\omega}^{(M-1)\omega} \frac{dx}{[x-(k-1)\omega]^{3n+4+2k}} &< \frac{1}{\omega}\int_{(m-1)\omega}^{\infty} \frac{dx}{\quad} \\ &= \frac{1}{\omega((m-k)\omega)^{3n+3+2k}}\frac{1}{3n+3+2k} \end{aligned} \tag{3.25}$$

so that our final bound is

$$|\,\sigma_{m,n,k}\,| \leqq \frac{3b\pi_n}{c_n}\frac{(3n+2+2k)!}{(3n+3)!}\frac{1}{\omega^{3n+4+2k}}\frac{1}{(m-k)^{3n+3+2k}} \tag{3.26}$$

4. The Interpolation Formulas. We will be concerned in this section with the two formulas

$$f'(x) = \frac{1}{\omega}\sum_{\lambda=-k}^{k} A_{k\lambda} f(x+\lambda\omega) + R \tag{4.1}$$

$$\int_0^{\omega} f(x)\,dx = \omega \sum_{\lambda=-k+1}^{k} B_{k\lambda} f(\lambda\omega) + R \tag{4.2}$$

where R represents the "error" (not necessarily the same in (4.1) and (4.2)). We must determine the coefficients $A_{k\lambda}$ and $B_{k\lambda}$ and also the error term.

The following notation is a modification of that used by Steffensen [7]. For a fixed positive quantity ω the operator ϵ is defined by $\epsilon^a f(x) = f(x + a\omega)$ and we also need the operators $\delta = (\epsilon^{\frac{1}{2}} - \epsilon^{-\frac{1}{2}})/\omega$, $\square = -\frac{1}{2}(\epsilon^{\frac{1}{2}} + \epsilon^{-\frac{1}{2}})$. Finally we define

$$x^{[0]} = 1,\ x^{[1]} = x,$$

$$x^{[2\nu]} = x^2(x^2 - \omega^2)\cdots(x^2 - (\nu-1)^2\omega^2)$$

$$x^{[2\nu+1]} = x\left(x^2 - \frac{\omega^2}{4}\right)\cdots\left(x^2 - \frac{(2\nu-1)^2}{4}\omega^2\right)$$

$$x^{[\nu]-j} = x^{[\nu]}/x^j$$

We take $1/(-1)!$ to mean zero.

The Derivative Formula. We begin with Stirling's formula

$$f(x) = \sum_{\nu=0}^{k}\left[\frac{x^{[2\nu]-1}}{(2\nu-1)!}\ \square\ \delta^{2\nu-1} + \frac{x^{[2\nu]}}{(2\nu)!}\,\delta^{2\nu}\right] f(0) + R, \tag{4.3}$$

$$R = \frac{x^{[2k+2]-1}}{(2k+1)!} f^{(2k+1)}(\xi), \qquad \min(x, -k\omega) \leqq \xi \leqq \max(x, k\omega) \tag{4.4}$$

This formula is proved in [7] for the case $\omega = 1$. It is proved for an arbitrary ω by replacing x by x/ω and the function $f(x)$ by the function $f(\omega x)$. Then

$$\begin{aligned} f'(0) &= \lim_{x\to 0}\frac{f(x)-f(0)}{x} = \lim_{x\to 0}\left[\sum_{\nu=1}^{k}\frac{x^{[2\nu]-2}}{(2\nu-1)!}\ \square\ \delta^{2\nu-1} f(0) + R\right] \\ &= \sum_{\nu=1}^{k}\frac{(-1)^{\nu-1}(\nu-1)!^2\,\omega^{2(\nu-1)}}{(2\nu-1)!}\ \square\ \delta^{2\nu-1} f(0) + R \qquad (4.5) \\ &= \sum_{\nu=0}^{k-1}\frac{(-1)^{\nu}\nu!^2\,\omega^{2\nu}}{(2\nu+1)!}\ \square\ \delta^{2\nu+1} f(0) + R \end{aligned}$$

where now

$$R = \frac{(-1)^k k!^2\,\omega^{2k}}{(2k+1)!} f^{(2k+1)}(\xi), \quad -k\omega \leqq \xi \leqq k\omega \tag{4.6}$$

We arrange (4.5) in a more usable form by expanding

$$\square\, \delta^{2\nu+1} = \frac{1}{2\omega^{2\nu+1}} (e^{\frac{1}{2}} + e^{-\frac{1}{2}}) (e^{\frac{1}{2}} - e^{-\frac{1}{2}})^{2\nu+1}$$

in powers of ϵ. The result is

$$\frac{1}{\omega^{2\nu+1}} (2\nu + 1)! \sum_i \frac{i - \nu}{(i+1)!\,(2\nu + 1 - i)!} (-1)^i e^{\nu - i}$$

Substituting into (4.5) we obtain

$$f'(0) = \omega^{-1} \sum_{\lambda=-k}^{k} A_{k\lambda} \epsilon^{\lambda} f(0) + R \tag{4.7}$$

with

$$A_{k\lambda} = \sum_{\nu=0}^{k-1} \sum_{i=-1}^{2\nu+1} - (-1)^{\nu-i} (\nu - i) \frac{\nu!^2}{(i+1)!(2\nu + 1 - i)!}$$

where only terms with $\nu - i = \lambda$ are included. Or

$$\begin{aligned} A_{k\lambda} &= -\lambda(-1)^{\lambda} \sum_{\nu=0}^{k-1} \frac{\nu!^2}{(\nu + 1 - \lambda)!\,(\nu + 1 + \lambda)!} \\ &= -\lambda(-1)^{\lambda} \sum_{\nu=|\lambda|}^{k} \frac{(\nu - 1)!}{(\nu - \lambda)!\,(\nu + \lambda)!}. \end{aligned} \tag{4.8}$$

It follows that $A_{k,-\lambda} = -A_{k,\lambda}$ and we may thus take $\lambda \geqq 0$.

Let us designate the sum appearing on the extreme right of (4.8) by $S_{k\lambda}$. I say that for $\lambda > 0$

$$S_{k\lambda} = \frac{1}{\lambda^2} \frac{k!^2}{(k + \lambda)!\,(k - \lambda)!}. \tag{4.9}$$

To prove (4.9) it suffices to show

$$1)\quad S_{k+1,\lambda} - S_{k,\lambda} = \frac{k!^2}{(k - \lambda + 1)!\,(k + \lambda + 1)!}, \qquad 2)\quad S_{\lambda\lambda} = \frac{(\lambda - 1)!}{(2\lambda)!},$$

where the S expressions mean the right side of (4.9). Both 1) and 2) follow by direct computation.

The formula (4.1) now follows immediately from (4.7) with the error term described by (4.6) and

$$A_{k\lambda} = -\frac{1}{\lambda} (-1)^{\lambda} \frac{k!^2}{(k + \lambda)!\,(k - \lambda)!}\ (\lambda \neq 0), \qquad A_{k0} = 0. \tag{4.10}$$

A table of $A_{k\lambda}$ is given in Appendix I.

The Integral Formula. We begin with Bessel's formula:

$$f\left(x + \frac{\omega}{2}\right) = \sum_{\nu=0}^{k-1} \left[\frac{x^{[2\nu+1]-1}}{(2\nu)!} \square\, \delta^{2\nu} e^{\frac{1}{2}} + \frac{x^{[2\nu+1]}}{(2\nu + 1)!} \delta^{2\nu+1} e^{\frac{1}{2}} \right] f(0) + R, \tag{4.11}$$

$$R = \frac{x^{[2k+1]-1}}{(2k)!} f^{(2k)}(\xi),$$

$$\min\left(x + \frac{\omega}{2}, -(k-1)\omega\right) \leqq \xi \leqq \max\left(x + \frac{\omega}{2}, k\omega\right).$$

This is likewise derived in [7] for the case $\omega = 1$ and can be modified as before.

We note that $\int_{-a}^{a} x^{[2\nu+1]}\,dx = 0$, as the integrand is an odd function, and we put

$$m_n = \frac{1}{n!}\int_{-(\omega/2)}^{\omega/2} x^{[n+1]-1}\,dx$$

and let M_n be this quantity when $\omega = 1$ (see [7], p. 109). It is easily shown that $m_{2\nu} = \omega^{2\nu+1}M_{2\nu}$. It now follows that

$$\int_0^\omega f(x)\,dx = \int_{-(\omega/2)}^{\omega/2} f\left(x + \frac{\omega}{2}\right)dx = \sum_{\nu=0}^{k-1} m_{2\nu} \square \delta^{2\nu}\epsilon^{\frac{1}{2}}f(0) + R$$

$$= \omega\sum_{\nu=0}^{k-1} M_{2\nu}\tfrac{1}{2}(1+\epsilon)(\epsilon^{\frac{1}{2}} - \epsilon^{-\frac{1}{2}})^{2\nu}f(0) + R$$

where now

$$R = \omega^{2k+1}M_{2k}f^{(2k)}(\xi), \qquad -(k-1)\omega \leqq \xi \leqq k\omega. \tag{4.12}$$

The term $(1+\epsilon)(\epsilon^{\frac{1}{2}} - \epsilon^{-\frac{1}{2}})^{2\nu}$, when expanded in powers of ϵ, ultimately gives

$$\sum_i \frac{(2\nu)!}{(i+1)!\,(2\nu-i)!}[1 - 2(\nu - i)](-1)^i\epsilon^{\nu-i}.$$

This leads to

$$\int_0^\omega f(x)\,dx = \omega\sum_{\lambda=-k+1}^{k} B_{k\lambda}\epsilon^{\lambda}f(0) + R \tag{4.13}$$

where

$$B_{k\lambda} = \tfrac{1}{2}(-1)^{\lambda}(1-2\lambda)\sum_{\nu=\{\lambda\}}^{k-1} M_{2\nu}\frac{(2\nu)!\,(-1)^{\nu}}{(\nu-\lambda+1)!\,(\nu+\lambda)!} \tag{4.14}$$

$$= \tfrac{1}{2}(-1)^{\lambda}(1-2\lambda)\int_{-\frac{1}{2}}^{\frac{1}{2}}\sum_{\nu=\{\lambda\}}^{k-1}(-1)^{\nu}\frac{x^{[2\nu+1]-1}}{(\nu-\lambda+1)!\,(\nu+\lambda)!}\,dx \tag{4.15}$$

and where $\{\lambda\} = \max(-\lambda, \lambda - 1)$. In (4.15) the ω implied in the $x^{[2\nu+1]-1}$ is understood to be 1. As $\{1-\lambda\} = \max(\lambda - 1, -\lambda) = \{\lambda\}$ and

$$1 - 2(1-\lambda) = -(1-2\lambda)$$

it follows from (4.14) that $B_{k,1-\lambda} = B_{k\lambda}$ so that we may suppose $\lambda > 0$.

Put

$$Q_{k\lambda}(x) = \sum_{\nu=\lambda}^{k} (-1)^{\nu} \frac{x^{[2\nu-1]-1}}{(\nu+\lambda-1)!\,(\nu-\lambda)!} \tag{4.16}$$

so that

$$B_{k\lambda} = (-1)^{\lambda} \tfrac{1}{2}(2\lambda-1) \int_{-\frac{1}{2}}^{\frac{1}{2}} Q_{k\lambda}(x)\, dx.$$

Then

$$Q_{k\lambda} = (-1)^{k} \frac{x^{[2k+1]-1}}{(k+\lambda-1)!\,(k-\lambda)!\left[x^2 - \dfrac{(2\lambda-1)^2}{4}\right]}. \tag{4.17}$$

We merely describe the steps in the proof which are carried out with some algebraic computation. Form the difference of right sides of (4.17) for k and $k-1$ and show that this reduces to the last term of the series in (4.16). The proof is then completed by showing that the right sides of (4.16) and (4.17) are equal when $k = \lambda$.

Thus

$$B_{k\lambda} = (-1)^{k+\lambda} \frac{2\lambda-1}{(k+\lambda-1)!\,(k-\lambda)!} \int_0^{\frac{1}{2}} \frac{x^{[2k+1]-1}}{x^2 - \dfrac{(2\lambda-1)^2}{4}}\, dx \qquad (\lambda > 0)$$

For computational purposes it is perhaps more convenient to use

$$B_{k\lambda} = (-1)^{k+\lambda} \frac{2\lambda-1}{(k+\lambda-1)!\,(k-\lambda)!} \frac{1}{2\cdot 4^{k-1}} \int_0^1 \frac{y^{[2k+1]-1}}{y^2 - (2\lambda-1)^2}\, dy \tag{4.18}$$

where $y^{[2k+1]-1}$ now means $(y^2-1^2)(y^2-2^2)\cdots(y^2-(2k-1)^2)$.

A table of the $B_{k\lambda}$ is given in Appendix II. (4.13) and (4.12) give us our sought formula.

5. Treatment of the Equations by the Interpolation Formulas. We now replace the derivative and integral occurring in the recurrence relation (2.10) for the r_n by their interpolatory expressions found in the last section. For this purpose the following notation is convenient.

$$r_{n,m} = r_n(m\omega), \qquad A_m = A(m\omega), \qquad B_m = B(m\omega).$$

Let M be a large integer (in this case, 1000) and take $\omega = 1/M$. Then we obtain

$$\begin{aligned} r_{n+1,m} = u_n \Big\{ & 3nA_m r_{n,m} - mA_m \sum_{\lambda=-k}^{k} A_{k\lambda} r_{n,m+\lambda} \\ & + m^{3n+3}\omega^4 \sum_{j=m}^{m-1} \sum_{\lambda=-k+1}^{k} B_{k\lambda} \frac{B_{\lambda+j}}{(\lambda+j)^{3n}} r_{n,\lambda+j} \Big\} + R_{m,n,k}. \end{aligned} \tag{5.1}$$

The error here is given by

$$R_{m,n,k} = u_n \left\{ -m\omega(-1)^k \frac{k!}{(2k+1)!}\, \omega^{2k} r_n^{(2k+1)}(\xi) + (m\omega)^{3n+3}\, \omega^{2c+1} M_{2k} \sum_{j=m}^{M-1} \left(\frac{B(\xi_j) r_n(\xi_j)}{\xi_j^{3n}} \right)^{(2k)} \right\} \qquad \{T\} \tag{5.2}$$

$$(m-k-1)\omega \leqq \xi \leqq (m+k+1)\omega, \tag{5.3}$$

$$(j-k+1)\omega \leqq \xi_j \leqq (k+1)\omega.$$

The double series of (5.1) may be arranged in the form

$$\sum_l C_{klm}(B_l/l^{3n}) r_{nl}$$

so that

$$C_{klm} = \sum_{\lambda=\lambda_1}^{\lambda_2} B_{k\lambda} \tag{5.4}$$

where $\lambda_1 = \max(-k+1, l-M+1)$, $\lambda_2 = \min(k, l-m)$

Thus the final formula for computation purposes may be written

$$r_{n+1,m} = u_n \left\{ 3nA_m r_{n,m} - m A_m \sum_{\lambda=-k}^{k} A_{k\lambda} r_{n,m+\lambda} + m^{3n+3} \omega^4 \sum_{l=m+k+1}^{M+k-1} C_{klm}(B_l/l^{3n}) r_{n,l} \right\} \tag{5.5}$$

It remains now to find a bound for $| R_{m,n,k} |$ so as to decide which value of k will maintain the errors within prescribed limits. As we are not interested in too large Mach numbers, we wish only the value for $m \geqq 250$.

Noting the appearance of $\sigma_{m,n,k}$ in (5.2), we make use of the inequality (3.26) as well as the bound of (3.23) for $| r_n^{(k)} |$. We also will use

$$| M_{2k} | \leqq [(2k)!/(k!)^2] 2^{-4k}$$

given in [7].

The combination of these inequalities leads directly to

$$| R_{m,n,k} | \leqq \frac{\pi_n}{C_{n+1}} \left[m \frac{k!}{(2k+1)!} \frac{(3n+k-1)!}{(3n-1)!} 2^{2k+1} \frac{1}{(m-k-1)^{2k+1}} + 3bm^{3n+3} \frac{(2k)!}{k!^2} \frac{(3n+2+2k)!}{(3n+3)!} 2^{-4k} \frac{1}{(m-k)^{3n+3+2k}} \right] \tag{5.6}$$

The components of the above which involve m can be simplified by means of the following inequalities:

$$\frac{m}{[m-(k+1)]^{2k+1}} = \frac{1}{[m-(k+1)]^{2k}} \frac{1}{1-\dfrac{k+1}{m}}$$

$$< \left(\frac{1}{m-(k+1)} \frac{1}{1-\dfrac{k+1}{m}} \right)^{2k} = \frac{1}{\left[m - 2(k+1) + \dfrac{(k+1)^2}{m} \right]^{2k}}$$

$$< \frac{1}{[m-2(k+1)]^{2k}}$$

$$\frac{m^{3n+3}}{(m-k)^{3n+3+2k}} = \frac{1}{\left(1-\dfrac{k}{m}\right)^{3n+3}} \frac{1}{(m-k)^{2k}} < \frac{1}{\left(1-\dfrac{k}{m}\right)^{3n+3}} \frac{1}{[m-2(k+1)]^{2k}}$$

Let us think of these inequalities utilized in (5.6) so that the factor

$$[m-(2k+1)]^{-3b}$$

may be removed from the bracket.

We shall now show that the first term of the bracket is small compared with the second. The ratio of these terms is

$$\left\{ \frac{k!^3\, 2^{2k+1}}{(2k+1)!(2k)!} \right\} \left\{ \frac{(3n+k-1)\cdots(3n)}{(3n+2+2k)\cdots(3n+4)} \right\} \left\{ \frac{\left(1-\dfrac{k}{m}\right)^{3n+3}}{3b} \right\}$$

We consider the three braces separately. Calling the first v_k we have

$$\frac{v_k}{v_{k-1}} = \frac{k^3 2^6}{(2k+1)(2k)^2(2k-1)} = \frac{16k}{4k^2-1}$$

For $k = 2, 3, 4$, this has the values $32/15, 48/35, 64/63 > 1$ while if $k \geqq 5$ it < 1 as then $4k^2 - 1 \geqq 20k - 1 > 16k$. Thus the maximum of v_k is attained for $k = 4$ and has the value $(4!)^3\, 2^{25}/9!8! = 31.7\cdots < 32$. The second brace cannot exceed $1/(3n+2+2k)\cdots(3n+4+k)$ and if we suppose $k \geqq 3$, $n \geqq 1$ this does not exceed $1/10\cdot 11 = 1/121$. The third brace does not exceed $1/3b$. We shall take $b = 17$ $(3b = 51 > 49)$ so an upper bound is $1/51$. Thus the entire ratio cannot exceed $32/121\cdot 51 < .0052$.

As $3b(1.0052) < 52$, we may drop the first term of (5.6) provided we replace $3b$ by 52 and $k > 2$.

Thus the final form for the general error is

$$|R_{m,n,k}| < \frac{\pi_n}{c_{n+1}} \frac{52}{[4m-8(k+1)]^{2k}} \frac{1}{\left(1-\dfrac{k}{m}\right)^{3n+3}} \frac{(2k)!}{k!^2} \frac{(3n+2+2k)!}{(3n+3)!}. \tag{5.7}$$

For practical purposes we make the restrictions $m \geqq 250$, $k \leqq 7$. Then $[4m-8(k+1)]^2 \geqq (1000-88)^2 > 831{,}000$, and $\left(1-\dfrac{k}{m}\right)^3 \geqq \left(1-\dfrac{7}{250}\right)^3 > .37$. Calling $R_{n,k}$ the value of the error for m thus restricted, we have

$$|R_{n,k}| < 52 \frac{\pi_n}{c_{n+1}} \frac{1}{(831{,}000)^k} \frac{1}{(.37)^{n+1}} \frac{(2k)!}{k!^2} \frac{(3n+2+2k)!}{(3n+3)!} \tag{5.8}$$

Tables to facilitate the use of this formula will be found in the appendix.

Computation of a number of particular values showed that $k = 5$ was a satisfactory order of interpolation for the accuracy desired. Some values of the error bound for this k are given in the next section.

6. Summary of the Results. The functions which are to be calculated on an automatic calculator of digiting type are $r_n(T)[n = 1, 2, \cdots]$. They are related to the ultimately desired $Q^{(n)}(\lambda)$, the functions used in the Bergman operators which generate stream functions of compressible fluids, by

$$Q^{(n)}(\lambda) = \frac{(-1)^n c_n}{(1-h^2)^n (1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1))} \frac{r_n(T)}{T^{3n}}$$

where

$$c_n = \frac{5}{12}\left(3 + \frac{5}{12(2)}\right)\left(3(2) + \frac{5}{12(3)}\right)\cdots\left(3(n-1) + \frac{5}{12n}\right)$$

The c_n are tabulated in Appendix III.

The formulas to be used in calculating the $r_n(T)$ are

$$r_{n+1,m} = \frac{1}{3n + \dfrac{5}{12(m+1)}}\left\{3nA_m r_{n,m} - mA_m \sum_{\lambda=-k}^{k} A_{k\lambda} r_{n,m+\lambda} + m^{3n+3}\omega^4 \sum_{j=m}^{M-1}\sum_{\lambda=-k+1}^{k} B_{k\lambda}\frac{B_{\lambda+j}}{(\lambda+j)^{3n}} r_{n,\lambda+j}\right\} \tag{6.1}$$

Here ω = the "span" or interval between computed values and will be taken to be .001.

$$r_{n,m} = r_n(m\omega), A_m = A(m\omega), B_m = B(m\omega).$$

$$A(T) = 1 - (1+h^2)T^2 + h^2T^4$$

$$B(T) = \frac{1}{4T^4(1-h^2T^2)}[5 - (1+6h^2)T^2 - (5-4h^2)T^4 + (1+2h^2)T^6]$$

h = a gas constant $\cong 6^{-1}$

k = half-order of the interpolation. The value to be used is $k = 5$.

M = total number of subdivisions (i.e. $M\omega = 1$). The value to be used is $M = 1000$.

$A_{k\lambda}$ and $B_{k\lambda}$ are the interpolation coefficients. They are tabulated in Appendices I and II.

Note 1. To calculate r_1, we consider an r_0 existing which $= 1$. Note that second term in brace in (6.1) vanishes.

Note 2. An alternative form for the double sum in (6.1) is

$$\sum_{l=m-k+1}^{M+k-1} C_{klm}(B_l/l^{3n}) r_{nl}$$

where

$$C_{klm} = \sum_{\lambda=\lambda_1}^{\lambda_2} B_{k\lambda}, \qquad \lambda_1 = \max(-k+1, l-M+1); \quad \lambda_2 = \min(k, l-m).$$

The Error. Calling the error that arises in the application of (6.1) in the interval $.25 \leqq T \leqq 1$ (or $250 \leqq m \leqq 1000$) $R_{n,k}$, we have the following bounds for it.

$$|R_{1,5}| < .0^{15}56, \; |R_{2,5}| < .0^{9}88, \; |R_{3,5}| < .0^{5}76$$
$$|R_{7,5}| < .0^{7}57, \; |R_{8,5}| < .0^{6}39, \; |R_{9,5}| < .0^{4}24$$
$$|R_{10,5}| < .0^{4}14, \; |R_{11,5}| < .0^{4}74, \; |R_{12,5}| < .0^{3}39$$

The error for $n = 5$, $k = 5$ and $T = .95$ $(m = 950) < .0^{11}13$

The formula for computing further error bounds will be found in Section V and tables to facilitate its use are given in Appendix III.

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APPENDIX I

Values of $A_{k\lambda}$ $(A_{k,-\lambda} = -A_{k,\lambda}, A_{k0} = 0)$

k \ λ	1	2	3	4	5	6	7
1	$\frac{1}{2}$						
2	$\frac{2}{3}$	$-\frac{1}{12}$					
3	$\frac{3}{4}$	$-\frac{3}{20}$	$\frac{1}{60}$				
4	$\frac{4}{5}$	$-\frac{1}{5}$	$\frac{4}{105}$	$-\frac{1}{280}$			
5	$\frac{5}{6}$	$-\frac{5}{21}$	$\frac{5}{84}$	$-\frac{5}{504}$	$\frac{1}{1260}$		
6	$\frac{6}{7}$	$-\frac{15}{56}$	$\frac{5}{63}$	$-\frac{1}{56}$	$\frac{1}{385}$	$-\frac{1}{5544}$	
7	$\frac{7}{8}$	$-\frac{7}{24}$	$\frac{7}{72}$	$-\frac{7}{264}$	$\frac{7}{1320}$	$-\frac{7}{10296}$	$\frac{1}{51480}$

APPENDIX II

Values of $B_{k\lambda}$ ($B_{k,1-\lambda} = B_{k,\lambda}$).

k \ λ	1	2	3	4	5	6	7
1	$\frac{1}{2}$						
2	$\frac{13}{24}$	$-\frac{1}{24}$					
3	$\frac{401}{720}$	$-\frac{31}{480}$	$\frac{11}{1440}$				
4	$\frac{68323}{120960}$	$-\frac{353}{4480}$	$\frac{1879}{120960}$	$-\frac{191}{120960}$			
5	$\frac{2067169}{3628800}$	$-\frac{40111}{453600}$	$\frac{581}{25920}$	$-\frac{28939}{7257600}$	$\frac{2497}{7257600}$		
6	$\frac{91473331}{159667200}$	$-\frac{15228847}{159667200}$	$\frac{600503}{21288960}$	$-\frac{6409423}{958003200}$	$\frac{331823}{319334400}$	$-\frac{73985}{958003200}$	
7	$\frac{250738892357}{435891456000}$	$-\frac{58483099261}{581188608000}$	$\frac{6918442211}{209227898880}$	$-\frac{3516373697}{373621248000}$	$\frac{579240401}{290594304000}$	$-\frac{1420656827}{5230697472000}$	$\frac{92427157}{5230697472000}$

APPENDIX III

Tables Used in the Error Terms

n	π_n	c_n	$\frac{\pi_n}{c_{n+1}}$	$52\frac{\pi_n}{c_{n+1}}\frac{1}{(.37)^{n+1}}$
1	17	.417	12.71	4870
2	189.9	1.338	24.2	25200
3	2410	8.21	32.2	89600
4	33500	74.8	37.0	278×10^3
5	542×10^3	903	39.8	809×10^3
6	1019×10^4	13610	41.4	228×10^4
7	220×10^6	246×10^3	42.5	631×10^4
8	539×10^7	518×10^4	43.3	175×10^5
9	1483×10^8	1245×10^5	44.1	478×10^5
10	452×10^{10}	337×10^7	44.7	132×10^6
11	1515×10^{11}	1011×10^8	45.4	359×10^6
12	555×10^{12}	334×10^{11}	46.1	986×10^6

k	$\frac{(2k)!}{(k!)^2}\frac{1}{(831{,}000)^k}$
4	147×10^{-24}
5	636×10^{-30}
6	280×10^{-35}
7	125×10^{-40}

THE SIMILARITY LAW OF TRANSONIC FLOW

By Theodore von Karman

Introduction.[1] The problem of the transonic flow around an obstacle has attracted the attention of many investigators. Both the theoretical and the experimental treatments of the question are difficult. The theoretical treatment is at the very least laborious because of the mixed character of the type of solution of the differential equations for compressible fluids when domains of subsonic and supersonic flow coexist. The experimental treatment is handicapped by the fact that wind tunnel tests become unreliable if the speed in the wind tunnel approaches the speed of sound.

The difficulty in wind tunnel tests arises from the fact that the fixed wall or free jet boundaries of the tunnel exert an unduly large influence on the flow about a model. The reason for this influence is indicated by the linearized differential equations for compressible fluids. Consider, for example, two-dimensional flow. The differential equation for the velocity potential ϕ of a slightly disturbed parallel flow of free stream Mach number M can be written:

$$(1 - M^2)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{1}$$

In this equation x and y are the coordinates parallel and normal, respectively, to the direction of the undisturbed uniform flow. For $M = 1$, Eq. (1) becomes

$$\frac{\partial^2\phi}{\partial y^2} = 0 \tag{2}$$

Hence, according to the linear perturbation theory every disturbance produced by an obstacle, for example an airfoil, at the speed of sound, extends with unchanged magnitude perpendicular to the flow direction.

Two conclusions can be drawn from this simple consideration; first, that at Mach number one the influence of boundaries at any finite distance must be very considerable; and second, that the perturbation theory based on Eq. (1) is not adequate to describe the phenomena in the neighborhood of the velocity of sound. Indeed, the drag of an airfoil or projectile in an ideal compressible fluid is given by this theory as zero if the Mach number is slightly smaller than unity and infinite if the Mach number is slightly larger than unity.

The present note considers the case where all flow velocities including free stream velocity differ only slightly from the velocity of sound, or more precisely, from the critical velocity of sound. The differences between the actual velocities and the critical sound velocity are considered as perturbation velocities. Whereas, however, the usual linearized theory neglects quadratic and higher terms of the perturbations, in the present case terms containing products of the perturbation velocities or their derivatives have to be retained. It will be seen, in fact,

[1] The similarity theory given in this paper was first presented in a seminar at the California Institute of Technology in May 1946 and later to the Sixth International Congress for Applied Mechanics, Paris, September 1946.

that at least one of these quadratic terms is of comparable magnitude to the linear term given by Eq. (2). One thereupon arrives at a simplified form of the differential equation for the velocity potential and at a similarity law for compressible flow patterns around similar aerodynamic shapes of different thickness ratios. This similarity law replaces the corresponding law deducible from the linearized differential equation (1), which in the subsonic case results in the Prandtl-Glauert rule and in the supersonic case in Ackeret's rule.

Two-dimensional Flow. Consider the differential equation for the potential in two-dimensional isentropic compressible flow,

$$\left(1 - \frac{u^2}{a^2}\right)\frac{\partial^2\phi}{\partial x^2} - 2\,\frac{uv}{a^2}\,\frac{\partial^2\phi}{\partial x\,\partial y} + \left(1 - \frac{v^2}{a^2}\right)\frac{\partial^2\phi}{\partial y^2} = 0 \tag{3}$$

in which u and v are the velocity components in the x and y directions and a is the velocity of sound at the speed $(u^2 + v^2)^{\frac{1}{2}}$, given by

$$a^2 + \frac{\gamma - 1}{2}\,(u^2 + v^2) = \frac{\gamma + 1}{2}\,a^{*2} \tag{4}$$

In (4), a^* is the so-called critical sound velocity, that is, the speed of sound at points where the local Mach number is unity.

In the linearized theory $1 - u^2/a^2$ in (3) is replaced by $1 - U^2/a^2$ where U is the free stream velocity. For this substitution to be approximately valid the perturbation velocity $u' = u - U$ must be small in comparison with the difference between flight velocity and sound velocity. This is evident from

$$a^2 - u^2 = a^2 - (U + u')^2 = a^2 - U^2 - 2Uu' - u'^2 \tag{5}$$

in which, at transonic speeds the neglected term $2U\,u'$ may well be of the same order as the term $a^2 - U^2$ which is retained in Eq. (1).

The perturbation analysis at transonic speeds may be carried out as follows. Defining the perturbation potential by

$$u = a^* + \frac{\partial\phi}{\partial x}, \qquad v = \frac{\partial\phi}{\partial y} \tag{6}$$

and using Eq. (4) one obtains, neglecting higher order terms,

$$\frac{u^2}{a^2} = 1 + \frac{(\gamma + 1)}{a^*}\,\frac{\partial\phi}{\partial x}, \qquad \frac{uv}{a^2} = \frac{1}{a^*}\,\frac{\partial\phi}{\partial y}, \qquad \frac{v^2}{a^2} = \frac{1}{a^{*2}}\left(\frac{\partial\phi}{\partial y}\right)^2. \tag{7}$$

Introducing (7) into (3) and neglecting terms of higher degree than the second in the derivatives of ϕ yields the following approximate differential equation

$$\frac{\partial^2\phi}{\partial y^2} = \frac{(\gamma + 1)}{a^*}\,\frac{\partial\phi}{\partial x}\,\frac{\partial^2\phi}{\partial x^2} + \frac{2}{a^*}\,\frac{\partial\phi}{\partial y}\,\frac{\partial^2\phi}{\partial x\,\partial y}. \tag{8}$$

Consider the boundary conditions: at infinity the velocity is equal to the undisturbed flow velocity U. Hence from Eq. (6)

$$\partial\phi/\partial x = U - a^* \tag{9}$$

and,

$$\partial\phi/\partial y = 0. \tag{10}$$

Using Eq. (4) and with $M_1 = U/a_2$ shows that Eq. (9) may be written as,

$$\frac{\partial\phi}{\partial x} = \frac{2}{(\gamma+1)}\, a^*(M_1 - 1). \tag{11}$$

At the airfoil the flow must follow the surface. Since the airfoil is thin one satisfies this condition on the x axis. Therefore

$$\text{at } y = \pm 0, \qquad \frac{1}{a^*}\frac{\partial\phi}{\partial y} = \frac{dy}{dx} = \tau h\left(\frac{x}{c}\right) \tag{12}$$

in which dy/dx, the slope of the airfoil surface at a chordwise location x, has been expressed as the product of the airfoil thickness ratio τ (τ = maximum thickness/airfoil chord c) and a function h of the order of magnitude unity.

Because of the assumption that all flow velocities are derived by small perturbations from the critical velocity a^*, we are interested in the limiting form of the solutions of Eq. (8) for small airfoil thickness ratios τ and small departure of the free stream velocity from sonic velocity. Evidently the solutions of Eq. (8) under these conditions could differ essentially from the solutions of the linearized Eq. (1) if at least two terms of (8), one linear and one quadratic, remained of the same order of magnitude as both τ and $M_1 - 1$ simultaneously approached zero. One obtains the limiting solution by introducing an affine transformation of the y coordinate, which has the effect of shrinking the flow pattern in the y direction in a definite way as both τ and $M_1 - 1$ approach zero. Such a shrinkage is necessary, for if $M_1 - 1$ approaches zero while τ remains finite, the streamlines remain equidistant from $y = 0$ to $y = \infty$.

To carry out this idea the following transformation to non-dimensional quantities is introduced:

$$\left.\begin{aligned} x &= c\xi, \qquad y = c(\tau\Gamma)^{-n}\eta \\ \phi &= ca^*\,\frac{(1 - M_1)}{\Gamma}\,F(\xi, \eta) \end{aligned}\right\} \tag{13}$$

in which

$$\Gamma = \tfrac{1}{2}(\gamma + 1) \tag{14}$$

is introduced into the transformation to permit the eventual inclusion in the similarity law of the effect of the physical properties of the fluid given by the specific heat ratio γ;[2] n is a number to be determined. The proportionality of ϕ to $1 - M_1$ is evidently necessary in accordance with the boundary condition at infinity (11).

[2] The rule for the influence of γ was found by W. Perl. I understand from private correspondence that C. Kaplan also extended my similarity considerations to variable values of γ.

The method exemplified by the preceding transformation is somewhat analogous to the method used in the hydrodynamic theory of the boundary layer. The boundary layer region shrinks to zero as the Reynolds Number R becomes infinitely great. The similarity law of the flow is thereupon obtained by an affine transformation which expands the thickness of the boundary layer region as $R \to \infty$. In the present case the affine transformation contracts the flow region as $M_1 \to 1$.

Substitution of Eqs. (13) and (14) into the potential equation (8) and the boundary conditions (10), (11), (12) yields, respectively,

$$\frac{\partial^2 F}{\partial \eta^2} = 2(\tau\Gamma)^{-2n}(1 - M_1)\frac{\partial F}{\partial \xi}\frac{\partial^2 F}{\partial \xi^2} + 2(1 - M_1)\frac{\partial F}{\partial \eta}\frac{\partial^2 F}{\partial \xi \partial \eta} \tag{15}$$

at ∞,

$$\partial F/\partial \xi = -1, \qquad \partial F/\partial \eta = 0, \tag{16}$$

at $\eta = \pm 0$,

$$\partial F/\partial \eta = (\tau\Gamma)^{1-n}(1 - M_1)^{-1}h(\xi). \tag{17}$$

The dependence of the preceding equations on τ, Γ and M_1 can be eliminated by setting

$$(\tau\Gamma)^{-2n}(1 - M_1) = K \tag{18}$$

$$(\tau\Gamma)^{1-n}(1 - M_1)^{-1} = K_1 \tag{19}$$

where K and K_1 are constants. Substitution of Eq. (18) in (19) shows that these equations can hold for arbitrary values of τ, Γ and M_1 only if

$$n = \tfrac{1}{2} \tag{20}$$

in which case

$$\frac{1 - M_1}{(\tau\Gamma)^{\frac{1}{2}}} = K, \tag{21}$$

and

$$K_1 = 1/K. \tag{22}$$

By virtue of Eq. (18), the last term on the right in Eq. (15) can be neglected and (15) becomes

$$\frac{\partial^2 F}{\partial \eta^2} = 2K\frac{\partial F}{\partial \xi}\frac{\partial^2 F}{\partial \xi^2}. \tag{23}$$

The boundary condition (17) becomes

$$\partial F/\partial \eta = 1/Kh(\xi). \tag{24}$$

The explicit dependence on K of Eqs. (23) and (24) can be shifted to the other boundary condition (16) by the substitution

$$f(\xi, \eta) = KF(\xi, \eta) \tag{25}$$

which gives in place of (23), (16) and (24), respectively,

$$\frac{\partial^2 f}{\partial \eta^2} = 2\frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \xi^2} \tag{26}$$

at ∞,

$$\partial f/\partial \xi = -K, \qquad \partial f/\partial \eta = 0 \tag{27}$$

at $\eta = \pm 0$,

$$\partial f/\partial \eta = h(\xi) \tag{28}$$

The transition to the physical plane, Eq. (13), is given by the relations

$$\left.\begin{aligned} x &= c\xi, \qquad y = c(\tau\Gamma)^{-1}\eta \\ \phi &= ca^* \frac{(1 - M_1)}{\Gamma} F(\xi, \eta) = ca^* \frac{\tau^{\frac{1}{2}}}{\Gamma^{\frac{1}{4}}} f(\xi, \eta) \end{aligned}\right\} \tag{29}$$

The relation (21) expresses the similarity law in the transonic case. A solution of Eqs. (26) to (28) for a given value of K and function $h(\xi)$ yields a whole class of flow patterns in accordance with (29), for similar airfoils of thickness ratios τ, at free stream Mach numbers M_1, and in fluids of specific heat ratio $\gamma = 2\Gamma - 1$ related by equation (21). The case $K > 0$ corresponds to subsonic free stream conditions; $K = 0$ corresponds to free stream velocity equal to sound velocity; $K < 0$ corresponds to supersonic free stream velocity, in which case the boundary condition (27) of undisturbed flow at infinity applies only ahead of the disturbance arising from the leading edge of the airfoil.

The solution of Eqs. (26) to (28) requires integration of the differential equation (26). This equation is simpler than the general differential equation (3). If it is linearized by application of a Legendre transformation, an equation of the form

$$\frac{\partial^2 \chi}{\partial u^2} = 2u\frac{\partial^2 \chi}{\partial v^2} \tag{30}$$

is obtained. This equation has solutions of the simple form

$$\chi = e^{\pm i\lambda v}\sqrt{u}\, Z(\tfrac{2}{3}\sqrt{2}\, i\lambda u^{3/2}) \tag{31}$$

where Z is a Bessel function of the order $\frac{1}{3}$.

Useful practical conclusions concerning the forces on airfoils in the transonic speed range can, however, be drawn from the similarity law (21) and the system of equations (26) to (28) without the necessity of an explicit solution of the latter Thus the pressure coefficient c_p on an airfoil in the approximation considered here is given by

$$c_p = \frac{p - p_1}{\frac{1}{2}\rho_1 U^2} \cong -\frac{2(u - U)}{U}. \tag{32}$$

Substitution of Eqs. (6) and (13) yields

$$c_p = -\frac{2(1 - M_1)}{\Gamma}\left(1 + \frac{\partial F}{\partial \xi}\right)$$

or,

$$c_p = -\frac{\tau^{\frac{1}{3}}}{\Gamma^{\frac{1}{3}}} \mathcal{P}\left[\frac{(\tau\Gamma)^{\frac{1}{3}}}{\sqrt{1 - M_1^2}}, \xi\right] \tag{33}$$

The Prandtl-Glauert rule is evidently obtained if $\mathcal{P}$ is a linear function of its argument.

The drag coefficient c_d of the airfoil is obtained by integrating the horizontal component of the pressure coefficient around the contour of the airfoil or

$$c_d = \frac{1}{c} \oint c_p \frac{dy}{dx} dx \tag{34}$$

In accordance with the representation of dy/dx as $\tau h(\xi)$, evidently the integrated result will be

$$c_d = \frac{\tau^{\frac{1}{3}}}{\Gamma^{\frac{1}{3}}} \mathcal{D}\left[\frac{(\tau\Gamma)^{\frac{1}{3}}}{\sqrt{1 - M_1^2}}\right] \tag{35}$$

where $\mathcal{D}$ is a function only of the quantity K as indicated. Similarly the lift coefficient c_l is given by

$$c_l = \frac{1}{c} \oint c_p \, dx$$

or,

$$c_l = \frac{\tau^{\frac{1}{3}}}{\Gamma^{\frac{1}{3}}} \mathcal{L}\left[\frac{(\tau\Gamma)^{\frac{1}{3}}}{\sqrt{1 - M_1^2}}\right]. \tag{36}$$

The moment coefficient is given by an expression similar to the preceding one with, however, the function $\mathcal{L}$ replaced by another function $\mathcal{M}$.

The preceding results (35) and (36) show, for example, that at the sonic velocity, for which the constant K is infinite, airfoils with similar thickness distributions have drag coefficients proportional to the $\frac{5}{3}$ power and lift coefficients proportional to the $\frac{2}{3}$ power of the thickness ratio, and that both coefficients are inversely proportional to the $\frac{1}{3}$ power of $(\gamma + 1)$. Furthermore, if a force coefficient is known experimentally for one airfoil as a function of Mach number in the transonic range, it is known by the preceding equations for similar airfoils of arbitrary but small thickness ratios in a similar speed range.

The theory presented here is based on the existence of a potential flow. It therefore supposes that the drag and lift are not affected in an essentially variable manner by viscosity in the range considered. It is known, however, that drag and lift in the transonic speed range are materially influenced by shock waves. The present theory should still apply, however, if the character of the interaction between shock wave and boundary layer does not change too radically in the range considered; for one can show that the conclusions of the present theory are still approximately valid if the shock waves which occur are approximately normal to the flow direction and if the vorticity beyond the shock waves is of small magnitude. It may be noted for example that the change in the value

small magnitude. It may be noted for example that the change in the value by the shock wave in the lower transonic regime (free stream velocity subsonic) is of the order of only a few per cent.

Axially symmetric flow. An analysis similar to that given for plane flows can likewise be given for axially symmetric flows. It will be sufficient to quote some of the corresponding results. The numbers at the left refer to the corresponding equations of the two-dimensional case. The simplified potential equation is:

$$(8) \qquad \frac{\partial^2\phi}{\partial y^2} + \frac{1}{y}\frac{\partial\phi}{\partial y} = \frac{(\gamma+1)}{a^*}\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2} + \frac{2}{a^*}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y} \qquad (37)$$

The boundary conditions are:

$$(9) \qquad \text{at } \infty, \quad \partial\phi/\partial x = U - a^*, \quad \partial\phi/\partial y = 0, \qquad (38)$$

$$(12) \qquad \text{at } y \to 0, \quad 2y\partial\phi/\partial y = U\, d(y^2)/dx = a^*\tau^2 cg(\xi). \qquad (39)$$

The affine transformation is:

$$(29) \qquad \left.\begin{aligned} x &= c\xi, \qquad y = \frac{c}{\tau\sqrt{\Gamma}}\eta, \\ \phi &= ca^*\frac{(1-M_1)}{\Gamma}F(\xi,\eta) = ca^*\tau^2 f(\xi,\eta). \end{aligned}\right\} \qquad (40)$$

The transformed potential equation and boundary conditions are:

$$(26) \qquad \frac{\partial^2 f}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial f}{\partial\eta} = 2\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\xi^2} \qquad (41)$$

$$(27) \qquad \text{at } \infty, \quad \partial f/\partial\xi = -K, \quad \partial f/\partial\eta = 0 \qquad (42)$$

$$(28) \qquad \text{at } \eta = 0, \quad 2\eta\,\partial f/\partial\eta = g(\xi). \qquad (43)$$

The similarity law is:

$$(21) \qquad \frac{1-M_1}{\tau^2\Gamma} = K \qquad (44)$$

The pressure and drag coefficients are:

$$(33) \qquad c_p = \tau^2\mathcal{P}\left[\left(\frac{\tau^2\Gamma}{1-M_1}\right), \xi\right] \qquad (45)$$

$$(35) \qquad c_d = \tau^2\mathcal{D}\left[\frac{\tau^2\Gamma}{(1-M_1)}\right]. \qquad (46)$$

Slightly supersonic flow. As has been mentioned, negative values of the quantity K, Eq. (21), correspond to supersonic free stream conditions. Now

several exact solutions of supersonic flow problems are known which, in the limit of small disturbance of a slightly supersonic free stream flow, can serve to check the present theory.

Consider first the supersonic flow against a two-dimensional wedge. For a given wedge vertex angle θ, there is a well-known least value of the supersonic free stream Mach number M_{1l} at which an oblique shock attached to the nose suffices to deflect the flow along the wedge surface. At smaller supersonic Mach numbers the flow pattern is altered radically with detachment of the shock and the existence of a local subsonic region near the nose. The Mach number M_{1l} is a function of the wedge angle θ. Inasmuch as M_{1l} separates two flow regimes of different character, it is to be expected that if similar flow patterns can exist near the nose of a wedge for different wedge angles θ, the corresponding free stream Mach numbers M_{1l} for the limiting shock would be related to θ by the similarity law. This expectation is indeed correct. The relation between M_{1l} and θ for small θ can be written[3]

$$\frac{M_{1l} - 1}{(\Gamma\theta)^{\frac{2}{3}}} = \tfrac{3}{4}(2)^{\frac{1}{3}} = 0.945 \tag{47}$$

This relation is of the form required by the similarity law (21), with, moreover, a value of the constant K of -0.945. An additional verification is given by the expression for the pressure on a wedge under conditions of small θ and slightly supersonic free stream Mach number M_1, where $M_1 > M_{1l}$. It is[4]

$$c_p = \frac{\Delta p}{\frac{1}{2}\rho U^2} = \frac{2\theta}{\sqrt{M_1^2 - 1}} + \frac{\Gamma\theta^2}{(M_1^2 - 1)^2}. \tag{48}$$

This expression can be obtained from Eq. (33) with a function $\mathcal{P}(K)$ of the form

$$\mathcal{P}(K) = \frac{1}{\sqrt{|K|}} + \frac{1}{4K^2}. \tag{49}$$

The similarity law (44) in the axially symmetric case can be checked in a similar manner from the results of Taylor and Maccoll[5] on the limiting Mach number for attached shock on cones.

Another known exact supersonic flow pattern is the well-known Prandtl-Meyer flow over a curved surface. It can be shown without difficulty that in expanding a flow from Mach number 1 to Mach number M by means of a Prandtl-Meyer deflection θ, we have

$$\frac{M - 1}{(\Gamma\theta)^{\frac{2}{3}}} = \frac{(3)^{\frac{1}{3}}}{2} = 1.04 \tag{50}$$

[3] von Karman, Th.: The Problem of Resistance in Compressible Fluids, Proc. Volta Congress, Rome, 1935, p. 239, Eq. (8.9).

[4] Loc. cit. Eq. (8.11).

[5] Maccoll, J. W.: The Conical Shock Wave formed by a Cone Moving at High Speed, Proc. Roy. Soc., vol. 159, pp. 459–472, 1937.

with a corresponding relation for the pressure coefficient

$$c_p = -3^{\frac{1}{3}} \frac{\theta^{\frac{1}{3}}}{\Gamma^{\frac{1}{3}}}. \tag{51}$$

These results agree respectively with the similarity law (21) and with Eq. (33) with a constant value for the function $\mathcal{P}$.

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TABLES FOR FACILITATING THE USE OF CHEBYSHEV'S QUADRATURE FORMULA

By Herbert E. Salzer

In a previous paper by the author[1], coefficients for finding the $(n-1)$th divided difference, were given for partial checking of the integrand in the n-point Gaussian quadrature formula. One of the drawbacks happened to be the very efficiency of the Gaussian formula, which precluded the possibility of applying to the limit any check equivalent to a differencing check. In the present case we consider the Chebyshev quadrature formula,[2]

$$\int_{-1}^{+1} f(z)\,dz = \frac{2}{n}\sum_{i=1}^{i=n} f(z_i) + R_n, \tag{1}$$

where

$$R_n = \int_{-1}^{+1} \frac{z^{n+1}}{(n+1)!} f^{(n+1)}(\xi)\,dz - \frac{2}{n(n+1)!}\sum_{i=1}^{i=n} z_i^{n+1} f^{(n+1)}(\xi_i),$$

and where ξ (which $= \xi(z)$ under the integral sign) satisfies $0 \le \xi \le z$, and $0 \le \xi_i \le z_i$, $i = 1, 2, \cdots, n$, and the z_i are the zeros of $T_n(z) \equiv$ polynomial part of $z^n \exp\left[\frac{-n}{2\cdot 3\cdot z^2} - \frac{n}{4\cdot 5\cdot z^4} - \frac{n}{6\cdot 7\cdot z^6} - \cdots\right]$. Here the advantage lies in that the weight coefficients are all equal, but for n points, Chebyshev's formula is exact for a polynomial of degree n (instead of $2n-1$). Now even though Chebyshev's quadrature formula does not possess the high efficiency of Gauss's formula, the integrand can be checked more thoroughly by a test equivalent to a differencing check, i.e. by taking the $(n-1)$th divided difference.

For a discussion of the meaning and value of the divided difference check, see the previous paper;[1] but for the sake of convenience the following is repeated here: 1) The divided difference is a linear function of the entries $f(z_i^{(n)})$, and since the integrand is taken at the same irregularly separated points, the coefficients of the entries $f(z_i^{(n)})$ are fixed quantities. 2) The analogue of the mth difference in the case of an irregularly spaced function is the mth divided difference $\times\, m! \times$ some "average value" of h^m. 3) A small $(n-1)$th divided difference of n entries represented by a polynomial of degree $n-2$ is in itself no guarantee as to their correctness; but an unduly large divided difference would indicate an error.

[1] "Coefficients for Facilitating the Use of the Gaussian Quadrature Formula," Jour. of Math. and Phys., 1946, pp. 244–246.

[2] This Chebyshev quadrature formula is not to be confused with another formula of the same name, also involving equally weighted coefficients, namely,

$$\int_{-1}^{+1} \frac{f(z)}{\sqrt{1-z^2}}\,dz = \frac{\pi}{n}\left[f\left(\cos\frac{\pi}{2n}\right) + f\left(\cos\frac{3\pi}{2n}\right) + \cdots + f\left(\cos\frac{(2n-1)\pi}{2n}\right)\right],$$

which is exact for $f(z)$ a polynomial of degree $2n-1$.

Even in this Chebyshev formula, the $(n-1)$th divided difference of n values of the integrand would be negligible only if the function behaved as a polynomial of degree $n-2$; whereas the use of n points in (1) applies with full accuracy to a polynomial up to degree n. This shows that a complete differencing check is not possible in the Chebyshev case; but this divided difference check will be more complete than in the Gaussian case. (Of course, in order to guarantee the correctness of an integrand as a polynomial of degree n, one might use the $(n+2)$-point Chebyshev formula, where the $(n+1)$th divided difference will be small if the integrand is correctly given as a polynomial of degree n.)

Chebyshev's formula is of practical use only when the roots of $T_n(z) = 0$ are all real. In other papers which give the roots as far as $n = 9$, the exceptional character of $n = 8$, which had complex roots, was always conspicuous; furthermore the statement was made by Ch. Jordan in his "Calculus of Finite Differences", p. 523, that $n = 10$ also had complex roots. Thus it was a question of great interest to determine whether there exists any $n > 9$ for which $T_n(z)$ would have all real zeros. This question was settled completely by S. Bernstein[3] who proved that for n other than 1–7 and 9, there is no equally weighted quadrature formula where the points $z_i^{(n)}$ are all real and within the interval $[-1, 1]$, and which is exact for any nth degree polynomial. Because of their general interests, the exact expressions for $T_n(z)$ for $n = 1$ to 12 are given below:

FIRST TWELVE POLYNOMIALS $T_n(z)$

$$T_1(z) = z, \quad 3T_2(z) = 3z^2 - 1, \quad 2T_3(z) = z(2z^2 - 1),$$
$$45T_4(z) = 45z^4 - 30z^2 + 1, \quad 72T_5(z) = z(72z^4 - 60z^2 + 7),$$
$$105T_6(z) = 105z^6 - 105z^4 + 21z^2 - 1$$
$$6480T_7(z) = z(6480z^6 - 7560z^4 + 2142z^2 - 149)$$
$$42525T_8(z) = 42525z^8 - 56700z^6 + 20790z^4 - 2220z^2 - 43$$
$$22400T_9(z) = z(22400z^8 - 33600z^6 + 15120z^4 - 2280z^2 + 53)$$
$$56133T_{10}(z) = 56133z^{10} - 93555z^8 + 49896z^6 - 9900z^4 + 561z^2 - 43$$
$$32659200T_{11}(z) = z(32659200z^{10} - 59875200z^8 + 369\,23040z^6 - 91\,63440z^4 + 817674z^2 - 33889)$$
$$7882875T_{12}(z) = 7882875z^{12} - 15765750z^{10} + 11036025z^8 - 3303300z^6 + 405405z^4 - 19110z^2 - 2161$$

The main feature of the table below is the checking coefficients $D_i^{(n)}$, so that in the n-point formula, after obtaining the integrand at the irregularly spaced points $z_i^{(n)}$, one can obtain the $(n-1)$th divided difference as $\sum_{i=1}^{i=n} D_i^{(n)} f(z_i^{(n)})$.

The quantities $D_i^{(n)}$ are tabulated for $n = 3(1)7, 9$ and $i = 1, 2, \cdots, n$, mostly to 9S; but specifically, for $n = 3$, exactly, for $n = 4$, to 8D, for $n = 5, 6, 7$, to 7D, and for $n = 9$, to 6D. The quantities $D_i^{(n)}$ are correct to about a unit in the last place that is given. Because the quantities $z_i^{(n)}$, the roots of $T_n(z) = 0$, are of fundamental importance, and since previous computations for the range $[-1, +1]$ gave either 8 decimal places only as far as $n = 5$, or only 5 decimals

[3] "Sur les Formules de Quadrature de Cotes et Tchebycheff" Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS. Nouvelle Serie, vol. XIV (1937), no. 6, pp. 323–326.

TABLE OF ROOTS $z_i^{(n)}$ AND CHECKING COEFFICIENTS $D_i^{(n)}$

$n = 3$

i	$z_i^{(3)}$	$D_i^{(3)}$
1	−.70710 67812	1.
2	.00000 00000	−2.
3	.70710 67812	1.

$n = 4$

	$z_i^{(4)}$	$D_i^{(4)}$
1	−.79465 44723	−1.05520 767
2	−.18759 24741	4.46993 141
3	.18759 24741	−4.46993 141
4	.79465 44723	1.05520 767

$n = 5$

i	$z_i^{(5)}$	$D_i^{(5)}$
1	−.83249 74870	1.30514 59
2	−.37454 14096	−6.44800 30
3	.00000 00000	10.28571 43
4	.37454 14096	−6.44800 30
5	.83249 74870	1.30514 59

$n = 6$

i	$z_i^{(6)}$	$D_i^{(6)}$
1	−.86624 68181	−1.48587 68
2	−.42251 86538	19.26271 31
3	−.26663 54015	−25.69696 10
4	.26663 54015	25.69696 10
5	.42251 86538	−19.26271 31
6	.86624 68181	1.48587 68

$n = 7$

i	$z_i^{(7)}$	$D_i^{(7)}$
1	−.88386 17008	1.89021 25
2	−.52965 67753	−20.27016 18
3	−.32391 18105	40.12491 58
4	.00000 00000	−43.48993 29
5	.32391 18105	40.12491 58
6	.52965 67753	−20.27016 18
7	.88386 17008	1.89021 25

$n = 9$

i	$z_i^{(9)}$	$D_i^{(9)}$
1	−.91158 93077	2.89337 3
2	−.60101 86554	−108.38009 1
3	−.52876 17831	158.03230 2
4	−.16790 61842	−263.86633 9
5	.00000 00000	422.64151 0
6	.16790 61842	−263.86633 9
7	.52876 17831	158.03230 2
8	.60101 86554	−108.38009 1
9	.91158 93077	2.89337 3

for $n = 2(1)7, 9$, the present table also includes the roots $z_i^{(n)}$ for $n = 3(1)7, 9$ and $i = 1, 2, \cdots, n$. The roots $z_i^{(n)}$ are given to 10 decimal places, correct to within a unit in the last place, thus superseding all previous calculations. (The case $n = 2$ is omitted, because there a divided difference check would be meaningless; for the sake of completeness one may note here the obvious fact that $z_1^{(2)} = -z_2^{(2)} = -\sqrt{1/3} = -.57735\ 02692$.)

For n odd, $D_i^{(n)} = D_{n-i+1}^{(n)}$, and for n even, $D_i^{(n)} = -D_{n-i+1}^{(n)}$. These relations were employed as a check upon the quantities $D_i^{(n)}$, all of which were computed independently. The middle coefficient for odd n was the only one that could not be checked that way. As an additional overall check upon the quantities $D_i^{(n)}$, they were employed to calculate the $(n-1)$th divided difference of $z_i^{(n)}$ for n even, and of $z_i^{(n)2}$ for n odd and >3, which, of course, must be zero. The roots $z_i^{(n)}$ were checked by substituting in the equation $T_n(z) = 0$.

It is important to notice that even though the Chebyshev quadrature formula is one where the errors are equally weighted, the checking process which uses the coefficient $D_i^{(n)}$, when applied to an integrand, say for $n = 9$, is much more likely to cause an error nearer to the center to stand out, because the coefficients nearer to the center are nearly 100 times larger.

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DETERMINATION OF A COMPRESSIBLE FLUID FLOW PAST AN OVAL-SHAPED OBSTACLE*

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1. Introduction. The fact that the mathematical theory of steady two-dimensional flows of an incompressible fluid (when formulated in an abstract manner) is essentially identical with the study of certain aspects of the theory of analytic functions of a complex variable makes possible the use of the highly developed methods of the latter theory.

We are, in physical applications, interested not merely in the determination of individual flow patterns but rather in the study of various properties of these flow patterns and of relations between these properties, in particular how certain quantities (like lift, drag, etc.) which are connected with the flow pattern depend upon various parameters (such as the form of the profile, the speed at infinity, etc.). Indeed, from the practical point of view the question of the actual determination of an *incompressible* flow pattern around a given body requires in many cases a prohibitive amount of computation, and only the recent development of computational devices promises to make possible in the not too distant future a satisfactory solution to the practical side of problems in conformal mapping which we meet in applications. Despite this situation the use of the theory of functions is of extreme importance in the theory of flows of an incompressible fluid.

Attempts to develop a mathematical theory of a compressible fluid along lines similar to those of an imcompressible fluid indicate the need for using computational devices to an even greater extent than in the case of an incompressible fluid. As is well known, the partial differential equation governing the stream-function ψ of a compressible fluid, considered as a function of the coordinates x, y of the physical plane (i.e., the plane in which the motion actually occurs), is non-linear. If, however, one considers ψ as a function of the components of the velocity vector or of certain related quantities (hodograph method), then the equation becomes linear and homogeneous. Naturally the investigation of a linear equation is considerably simpler than that of a non-linear one; in this case the superposition principle holds, and recent developments in the theory of linear partial differential equations make it possible to generalize many methods of the theory of analytic functions. On the other hand the use of the hodograph method requires (even in the incompressible case) a considerable amount of computational work.

This situation makes it desirable to prepare once and for all certain tables which will enable one to determine in a comparatively short time flow patterns (in the physical plane) which are of interest, and in a form in which one can see,

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at least to a certain extent, the dependence of the quantities characteristic of the flow pattern on the form of the body, the speed at infinity, etc. Such tables can be prepared conveniently by the use of modern computational machinery. As a guide for the preparation of such tables it is useful to have some illustrative examples for the determination of flow patterns.

The essential part of such tables in the case of integral operator method will consist of tables of certain auxiliary functions $Q^{(n)}$, $q^{(n,\kappa)}$, $\kappa = 1, 2, n = 1, 2, \cdots$ which are described in detail in [2, §11; 6, §1; 8, §§6, 7].[1]

In the present paper the first six $Q^{(n)}$ have been tabulated but in order to obtain greater accuracy in computing the streamfunction in a reasonably extended domain of the hodograph plane a larger number of the $Q^{(n)}$ and the $q^{(n,\kappa)}$, $\kappa = 1, 2$, are needed. Since the aim of the present paper is an illustration of the integral operator method, we have restricted ourselves to an example with a comparatively small Mach number and to a flow pattern obtained by applying the operator to a comparatively simple analytic function. In the case of higher speeds, the procedure is exactly the same, but requires a knowledge of more $Q^{(n)}$ as well as $q^{(n,\kappa)}$. We defer this treatment, therefore, until a larger number of them is available. Then we will be able to compute, with the same amount of work[2] various flow patterns with a sufficient degree of accuracy.

In [3], [4] a computation has been carried out for a body of nearly circular shape, and in the present paper in §2–§4 we consider an obstacle of nearly elliptical form. In §5 we make some additional remarks. The authors wish to express their gratitude to R. von Mises and M. Schiffer for their helpful advice n connection with this paper.

2. The principle of the tables to be prepared.[3] As had been shown in [1] and [2], a "general formula" for the streamfunction of a compressible fluid flow can be written in an implicit form, as follows:

$$(2.1)\quad \psi = \sum A_\nu \psi_\nu(\lambda, \theta), \qquad x = \sum A_\nu X_\nu(\lambda, \theta), \qquad y = \sum A_\nu Y_\nu(\lambda, \theta).$$

Here x, y are cartesian coordinates of the plane in which the flow occurs (physical plane), the A_ν are arbitrary constants, λ, θ are certain parametric variables (see

[1] Numbers in brackets refer to the bibliography. The present paper does not presuppose the knowledge of previous publications.

[2] These functions can be used for the determination of partially supersonic flows as well. It seems that the difficulties encountered in applying the hodograph method, are only partially of purely mathematical nature. An essential bar in this procedure is that every step requires very long computations. In particular, after the streamfunction is already defined in the hodograph plane, the transition to the physical plane is exceedingly time consuming. It should be therefore very useful to develop certain methods so that operators of special computational devices could carry out this work in the most economical manner.

[3] The understanding of the present section requires the knowledge of the mathematical theory of the hodograph method and the method of integral operators. In particular we make use here of the fact that hodographs of flow patterns are situated on Riemann's surfaces. See for details [6]. The contents of this section are not essential for understanding of the remainder of the paper.

TABLE I

The values of T, M, λ, H and $dH/d\lambda$, ($k = 1.405$)

T	M	λ	H	$dH/d\lambda$	T	M	λ	H	$dH/d\lambda$
.10	.9950	−.00028	2.5236	1487.1					
.11	.9939	−.00037	2.4074	1061.3	.51	.8602	−.04509	1.1786	2.9245
.12	.9927	−.00048	2.3059	779.39	.52	.8542	−.04823	1.1699	2.6628
.13	.9915	−.00062	2.2165	586.26	.53	.8480	−.05154	1.1614	2.4254
.14	.9902	−.00077	2.1372	450.12	.54	.8417	−.05505	1.1532	2.2098
.15	.9887	−.00096	2.0660	351.67	.55	.8352	−.05875	1.1455	2.0140
.16	.9871	−.00116	2.0017	278.98	.56	.8285	−.06265	1.1380	1.8357
.17	.9854	−.00139	1.9433	224.28	.57	.8216	−.06678	1.1307	1.6731
.18	.9837	−.00165	1.8899	182.42	.58	.8146	−.07113	1.1239	1.5252
.19	.9818	−.00195	1.8408	149.67	.59	.8074	−.07572	1.1171	1.3898
.20	.9798	−.00229	1.7957	124.38	.60	.8000	−.08056	1.1107	1.2664
.21	.9777	−.00265	1.7539	104.05	.61	.7924	−.08567	1.1045	1.1534
.22	.9755	−.00306	1.7152	87.702	.62	.7846	−.09108	1.0985	1.0502
.23	.9732	−.00350	1.6790	74.425	.63	.7766	−.09678	1.0929	.95582
.24	.9708	−.00400	1.6454	63.557	.64	.7684	−.10282	1.0873	.86927
.25	.9682	−.00453	1.6138	54.579	.65	.7599	−.10919	1.0821	.79010
.26	.9656	−.00512	1.5841	47.110	.66	.7513	−.11593	1.0770	.71749
.27	.9629	−.00576	1.5563	40.862	.67	.7424	−.12305	1.0621	.65091
.28	.9600	−.00644	1.5300	35.592	.68	.7332	−.13059	1.0674	.58989
.29	.9570	−.00719	1.5051	31.125	.69	.7238	−.13858	1.0629	.53395
.30	.9539	−.00799	1.4817	27.324	.70	.7141	−.14703	1.0586	.48266
.31	.9507	−.00887	1.4595	24.066	.71	.7042	−.15600	1.0546	.43568
.32	.9474	−.00978	1.4383	21.264	.72	.6940	−.16553	1.0506	.39253
.33	.9440	−.01080	1.4185	18.845	.73	.6834	−.17564	1.0469	.35303
.34	.9404	−.01186	1.3994	16.744	.74	.6726	−.18638	1.0435	.31689
.35	.9367	−.01301	1.3812	14.914	.75	.6614	−.19783	1.0398	.28365
.36	.9330	−.01424	1.3639	13.316	.76	.6499	−.21004	1.0366	.25332
.37	.9290	−.01555	1.3475	11.915	.77	.6380	−.22305	1.0335	.22557
.38	.9250	−.01693	1.3317	10.682	.78	.6258	−.23697	1.0305	.20022
.39	.9208	−.01843	1.3171	9.5978	.79	.6131	−.25188	1.0277	.17709
.40	.9165	−.02000	1.3024	8.6333	.80	.6000	−.26785	1.0250	.15599
.41	.9121	−.02168	1.2886	7.7794	.81	.5864	−.28506	1.0225	.13682
.42	.9075	−.02346	1.2755	7.0214	.82	.5724	−.30358	1.0202	.11938
.43	.9028	−.02535	1.2628	6.3447	.83	.5578	−.32362	1.0179	.10359
.44	.8980	−.02736	1.2508	5.7409	.84	.5426	−.34531	1.0158	.08931
.45	.8930	−.02949	1.2391	5.1998	.85	.5268	−.36892	1.0139	.07645
.46	.8879	−.03174	1.2280	4.7149	.86	.5103	−.39472	1.0121	.06488
.47	.8827	−.03411	1.2173	4.2791	.87	.4931	−.52301	1.0104	.05452
.48	.8773	−.03663	1.2071	3.8869	.88	.4750	−.45422	1.0088	.04530
.49	.8717	−.03930	1.1972	3.5329	.89	.4560	−.48885	1.0074	.03713
.50	.8660	−.04212	1.1877	3.2134	.90	.4359	−.52759	1.0061	.02995
					.91	.4146	−.57130	1.0049	.02370
					.92	.3919	−.62116	1.0039	.01829
					.93	.3676	−.67883	1.0029	.01368
					.94	.3412	−.74677	1.0022	.00983
					.95	.3122	−.82872	1.0015	.00668
					.96	.2800	−.93105	1.0009	.00418
					.97	.2431	−1.06557	1.0005	.00230
					.98	.1990	−1.25893	1.0003	.00100
					.99	.1411	−1.59606	1.0001	.00025
					1.00	.0000	−∞	1.0000	0

TABLE II. 1

Values of $Q^{(n)}$, $n = 1, 2, 3, 4, 5, 6$ ($k = 1.405$)

T	M	$Q^{(1)}$	$Q^{(2)}$	$Q^{(3)}$	$Q^{(4)}$	$Q^{(5)}$	$Q^{(6)}$
400	9165	−6.0454819 00	1.1910664 02	−2.3483866 03	5.4202450 04	−1.2883204 06	2.8113334 07
401	9161	−5.9905800 00	1.1714567 02	−2.2915454 03	5.2461467 04	−1.2368527 06	2.6772556 07
402	9156	−5.9362471 00	1.1522052 02	−2.2361896 03	5.0779732 04	−1.1875387 06	2.5498237 07
403	9152	−5.8824753 00	1.1333051 02	−2.1822769 03	4.9155101 04	−1.1402840 06	2.4286972 07
404	9148	−5.8292583 00	1.1147490 02	−2.1297671 03	4.7585533 04	−1.0949985 06	2.3135501 07
405	9143	−5.7765889 00	1.0965300 02	−2.0786209 03	4.6069057 04	−1.0515967 06	2.2040780 07
406	9139	−5.7244615 00	1.0786414 02	−2.0287997 03	4.4603754 04	−1.0099961 06	2.0999887 07
407	9134	−5.6728678 00	1.0610763 02	−1.9802670 03	4.3187817 04	−9.7011885 05	2.0010073 07
408	9130	−5.6218023 00	1.0438283 02	−1.9329861 03	4.1819458 04	−9.3188961 05	1.9068732 07
409	9126	−5.5712582 00	1.0268910 02	−1.8869225 03	4.0497008 04	−8.9523768 05	1.8173403 07
410	9121	−5.5212289 00	1.0102583 02	−1.8420428 03	3.9218819 04	−8.6009451 05	1.7321746 07
411	9116	−5.4717072 00	9.9392389 01	−1.7983134 03	3.7983333 04	−8.2639528 05	1.6511549 07
412	9112	−5.4226900 00	9.7788204 01	−1.7557037 03	3.6789048 04	−7.9407829 05	1.5740725 07
413	9107	−5.3741669 00	9.6212667 01	−1.7141816 03	3.5634489 04	−7.6308349 05	1.5007271 07
414	9103	−5.3261354 00	9.4665221 01	−1.6737177 03	3.4518275 04	−7.3335459 05	1.4309314 07
415	9098	−5.2785880 00	9.3145332 01	−1.6342834 03	3.3439063 04	−7.0483807 05	1.3645080 07
416	9094	−5.2315196 00	9.1652417 01	−1.5958498 03	3.2395538 04	−6.7748133 05	1.2726357 07
417	9089	−5.1849223 00	9.0185930 01	−1.5583888 03	3.1386440 04	−6.5123500 05	1.2411044 07
418	9084	−5.1387932 00	8.8745407 01	−1.5218756 03	3.0410608 04	−6.2605274 05	1.1838141 07
419	9079	−5.0931254 00	8.7330292 01	−1.4862831 03	2.9466848 04	−6.0188872 05	1.1292471 07
420	9075	−5.0479139 00	8.5940131 01	−1.4515874 03	2.8554075 04	−5.7870055 05	1.0773336 07
421	9071	−5.0031515 00	8.4574375 01	−1.4177631 03	2.7671177 04	−5.5644616 05	1.0278773 07
422	9066	−4.9588348 00	8.3232580 01	−1.3847867 03	2.6817141 04	−5.3508675 05	9.8077744 06
423	9061	−4.9149583 00	8.1914299 01	−1.3526363 03	2.5990976 04	−5.1458483 05	9.3591117 06
424	9057	−4.8715167 00	8.0619050 01	−1.3212887 03	2.5191711 04	−4.9490416 05	8.9318908 06
825	9052	−4.8285038 00	7.9346379 01	−1.2907225 03	2.4418414 04	−4.7601012 05	8.5248362 06
426	9047	−4.7859143 00	7.8095846 01	−1.2609164 03	2.3670188 04	−4.5787002 05	8.1370308 06
427	9043	−4.7437455 00	7.6867063 01	−1.2318512 03	2.2946208 04	−4.4045281 05	8.6145909 06
428	9038	−4.7019907 00	7.5659561 01	−1.2035058 03	2.2245604 04	−4.2372772 05	7.4154497 06
429	9033	−4.6606446 00	7.4472958 01	−1.1758616 03	2.1567595 04	−4.0766624 05	7.0799169 06
430	9028	−4.6197048 00	7.3306857 01	−1.1489002 03	2.0911418 04	−3.9224112 05	6.7601392 06
431	9024	−4.5791637 00	7.2160841 01	−1.1226028 03	2.0276318 04	−3.7742575 05	6.4553405 06
432	9019	−4.5390187 00	7.1034554 01	−1.0969523 03	1.9661588 04	−3.6319502 05	6.1647989 06
433	9014	−4.4992640 00	6.9927587 01	−1.0719311 03	1.9066530 04	−3.4952469 05	5.8878178 06
434	9009	−4.4598962 00	6.8839621 01	−1.0475236 03	1.8490493 04	−3.3639208 05	5.6237477 06
435	9004	−4.4209098 00	6.7770238 01	−1.0237122 03	1.7932814 04	−3.2377457 05	5.3719558 06
436	8999	−4.3822999 00	6.6719111 01	−1.0004823 03	1.7392888 04	−3.1165131 05	5.1318545 06

437	8995	−4.3440641	00	6.5685891	01	−9.7781800	02	1.6870112	04	−3.0000198	05	4.9028775	06
438	8990	−4.3061958	00	6.4670256	01	−9.5570503	02	1.6363921	04	−2.8880745	05	4.6844959	06
439	8985	−4.2686930	00	6.3671859	01	−9.3412860	02	1.5873752	04	−2.7804892	05	4.4761972	06
440	8980	−4.2315508	00	6.2690387	01	−9.1307509	02	1.5399075	04	−2.6770898	05	4.2775027	06
441	8975	−4.1947640	00	6.1725501	01	−8.9253033	02	1.4939363	04	−2.5777023	05	4.0879474	06
442	8970	−4.1583290	00	6.0776899	01	−8.7248126	02	1.4494119	04	−2.4821654	05	3.9070966	06
443	8965	−4.1222428	00	5.9844288	01	−8.5291539	02	1.4062875	04	−2.3903243	05	3.7345403	06
444	8960	−4.0865010	00	5.8927378	01	−8.3382030	02	1.3645164	04	−2.3020314	05	3.5698832	06
445	8955	−4.0510984	00	5.8025836	01	−8.1518296	02	1.3240521	04	−2.2171388	05	3.4127448	06
446	8950	−4.0160332	00	5.7139409	01	−7.9699241	02	1.2848531	04	−2.1355137	05	3.2627763	06
447	8945	−3.9813001	00	5.6267812	01	−7.7923677	02	1.2468770	04	−2.0570231	05	3.1196152	06
448	8940	−3.9468962	00	5.5410761	01	−7.6190495	02	1.2100838	04	−1.9815421	05	2.9829968	06
449	8935	−3.9128172	00	5.4567997	01	−7.4498600	02	1.1744343	04	−1.9089493	05	2.8525588	06
450	8930	−3.8790594	00	5.3739244	01	−7.2846950	02	1.1398911	04	−1.8391300	05	2.7280281	06
451	8925	−3.8456205	00	5.2924264	01	−7.1234512	02	1.1064183	04	−1.7719741	05	2.6091293	06
452	8920	−3.8124958	00	5.2122792	01	−6.9660293	02	1.0739808	04	−1.7073753	05	2.4970399	06
453	8915	−3.7796814	00	5.1334568	01	−6.8123304	02	1.0425444	04	−1.6452312	05	2.3871795	06
454	8910	−3.7471743	00	5.0559380	01	−6.6622630	02	1.0120773	04	−1.5836265	05	2.2835782	06
455	8905	−3.7149727	00	4.9796966	01	−6.5157310	02	9.8254730	03	−1.5279243	05	2.1847496	06
456	8900	−3.6830702	00	4.9047103	01	−6.3726483	02	9.5392449	03	−1.4725788	05	2.0902927	06
457	8895	−3.6514658	00	4.8309556	01	−6.2329261	02	9.2617903	03	−1.4193219	05	2.0000628	06
458	8890	−3.6201554	00	4.7584103	01	−6.0964794	02	8.9928276	03	−1.3680720	05	1.9138646	06
459	8884	−3.5891354	00	4.6870526	01	−5.9632268	02	8.7320847	03	−1.3187508	05	1.8315112	06
460	8879	−3.5584040	00	4.6168611	01	−5.8330881	02	8.4792939	03	−1.2712819	05	1.7528256	06
461	8874	−3.5279569	00	4.5478163	01	−5.7059867	02	8.2342050	03	−1.2255936	05	1.6776388	06
462	8869	−3.4977908	00	4.4798951	01	−5.5818445	02	7.9965665	03	−1.1816155	05	1.6057890	06
463	8864	−3.4679034	00	4.4130776	01	−5.4605855	02	7.7661362	03	−1.1392803	05	1.5371218	06
464	8858	−3.4382916	00	4.3473457	01	−5.3421431	02	7.5426916	03	−1.0985255	05	1.4714934	06
465	8853	−3.4089514	00	4.2826780	01	−5.2264428	02	7.3260037	03	−1.0592884	05	1.4087631	06
466	8848	−3.3798809	00	4.2190563	01	−5.1134191	02	7.1158599	03	−1.0215106	05	1.3487991	06
467	8843	−3.3510770	00	4.1564626	01	−5.0030055	02	6.9120529	03	−9.8513592	04	1.2914753	06
468	8837	−3.3225367	00	4.0948776	01	−4.8951361	02	6.7143794	03	−9.5010929	04	1.2366709	06
469	8832	−3.2942577	00	4.0342850	01	−4.7897509	02	6.5226495	03	−9.1637956	04	1.1842721	06
470	8827	−3.2662363	00	3.9746650	01	−4.6867844	02	6.3366672	03	−8.8389555	04	1.1341682	06

TABLE II. 2

Values of $Q^{(n)}$, $n = 1, 2, 3, 4, 5, 6$ ($k = 1.405$)

T	M	$Q^{(1)}$	$Q^{(2)}$	$Q^{(3)}$	$Q^{(4)}$	$Q^{(5)}$	$Q^{(6)}$
471	8821	−3.2384701 00	3.9160015 01	−4.5861801 02	6.1562582 03	−8.5261009 04	1.0862565 06
472	8816	−3.2109562 00	3.8582768 01	−4.4878780 02	5.9812412 03	−8.2247676 04	1.0404369 06
473	8811	−3.1836919 00	3.8014747 01	−4.3918229 02	5.8114481 03	−7.9345131 04	9.9661480 05
474	8805	−3.1566747 00	3.7455796 01	−4.2979592 02	5.6467176 03	−7.6549156 04	9.5470071 05
475	8800	−3.1299030 00	3.6905763 01	−4.2062367 02	5.4868927 03	−7.3855744 04	9.1460962 05
476	8794	−3.1033720 00	3.6364462 01	−4.1165971 02	5.3318109 03	−7.1260819 04	8.7625695 05
477	8789	−3.0770811 00	3.5831776 01	−4.0289955 02	5.1813341 03	−6.8760788 04	8.3956739 05
478	8784	−3.0510267 00	3.5307523 01	−3.9433776 02	5.0353103 03	−6.6351910 04	8.0446427 05
479	8778	−3.0252072 00	3.4791580 01	−3.8596998 02	4.8936074 03	−6.4030833 04	7.7087823 05
480	8773	−2.9996187 00	3.4283778 01	−3.7779101 02	4.7560842 03	−6.1794115 04	7.3873994 05
481	8767	−2.9742603 00	3.3784003 01	−3.6979679 02	4.6226191 03	−5.9638682 04	7.0798641 05
482	8762	−2.9491286 00	3.3292095 01	−3.6198250 02	4.4930788 03	−5.7561367 04	6.7855497 05
483	8756	−2.9242219 00	3.2807927 01	−3.5434403 02	4.3673471 03	−5.5559284 04	6.5038735 05
484	8751	−2.8995371 00	3.2331362 01	−3.4687698 02	4.2453031 03	−5.3629551 04	6.2342685 05
485	8745	−2.8750727 00	3.1862270 01	−3.3957738 02	4.1268349 03	−5.1769484 04	5.9762075 05
486	8740	−2.8508259 00	3.1400523 01	−3.3244115 02	4.0118319 03	−4.9976442 04	5.7291736 05
487	8734	−2.8267941 00	3.0945989 01	−3.2546429 02	3.9001861 03	−4.8247921 04	5.4926815 05
488	8728	−2.8029758 00	3.0498550 01	−3.1864308 02	3.7917962 03	−4.6581492 04	5.2662656 05
489	8723	−2.7793688 00	3.0058080 01	−3.1197397 02	3.6865640 03	−4.4974885 04	5.0494880 05
490	8717	−2.7559704 00	2.9624470 01	−3.0545299 02	3.5843883 03	−4.3425791 04	4.8419151 05
491	8711	−2.7327788 00	2.9197600 01	−2.9907699 02	3.4851807 03	−4.1932142 04	4.6431532 05
492	8706	−2.7097914 00	2.8777339 01	−2.9284223 02	3.3888471 03	−4.0491817 04	4.4528059 05
493	8700	−2.6870073 00	2.8363597 01	−2.8674562 02	3.2953027 03	−3.9102891 04	4.2705157 05
494	8695	−2.6644231 00	2.7956243 01	−2.8078367 02	3.2044599 03	−3.7763387 04	4.0959207 05
495	8689	−2.6420370 00	2.7555180 01	−2.7495330 02	3.1162388 03	−3.6471528 04	3.9286883 05
496	8683	−2.6198474 00	2.7160303 01	−2.6925142 02	3.0305596 03	−3.5225545 04	3.7685008 05
497	8678	−2.5978523 00	2.6771502 01	−2.6367499 02	2.9473451 03	−3.4023741 04	3.6150500 05
498	8672	−2.5760495 00	2.6388681 01	−2.5822115 02	2.8665215 03	−3.2864488 04	3.4680432 05
499	8666	−2.5544375 00	2.6011731 01	−2.5288688 02	2.7880163 03	−3.1746225 04	3.3272007 05
500	8660	−2.5330136 00	2.5640558 01	−2.4766949 02	2.7117598 03	−3.0667438 04	3.1922569 05
510	8602	−2.3287347 00	2.2226057 01	−2.0135316 02	2.0590008 03	−2.1751981 04	2.1163192 05
520	8542	−2.1413499 00	1.9293023 01	−1.6410729 02	1.5686776 03	−1.5494599 04	1.4103421 05
530	8480	−1.9692741 00	1.6768238 01	−1.3406307 02	1.1989175 03	−1.1081706 04	9.4448453 04
540	8417	−1.8110959 00	1.4590529 01	−1.0975710 02	9.1903574 02	−7.9554578 03	6.3541051 04
550	8352	−1.6655556 00	1.2708651 01	−9.0038907 01	7.0643999 02	−5.7312283 03	4.2931511 04
560	8285	−1.5315294 00	1.1079548 01	−7.4000837 01	5.4441896 02	−4.1423786 03	2.9123026 04
570	8216	−1.4080066 00	9.6669147 00	−6.0923795 01	4.2055489 02	−3.0030924 03	1.9829571 04

580	8146	−1.2940830 00	8.4400784 00	−5.0236255 01	3.2558320 02	−2.1832524 03	1.3548403 04
590	8074	−1.1889441 00	7.3730640 00	−4.1482550 01	2.5256428 02	−1.5913225 03	9.2863671 03
600	8000	−1.0918558 00	6.4437856 00	−3.4297959 01	1.9627845 02	−1.1626093 03	6.3836770 03
610	7924	−1.0021558 00	5.6334548 00	−2.8389896 01	1.5278689 02	−8.5120716 02	4.3999750 03
620	7846	−9.192462 −01	4.9260288 00	−2.3522805 01	1.1910609 02	−6.2440652 02	3.0399841 03
630	7766	−8.425843 −01	4.3077902 00	−1.9506609 01	9.2969194 01	−4.5881182 02	2.1048515 03
640	7684	−7.716784 −01	3.7669734 00	−1.6187401 01	7.2647774 01	−3.3763213 02	1.4601268 03
650	7599	−7.060818 −01	3.2934772 00	−1.3440331 01	5.6820475 01	−2.4876973 02	1.0145273 03
660	7513	−6.453885 −01	2.8786068 00	−1.1163839 01	4.4793678 01	−1.8347437 02	7.0595563 02
670	7424	−5.892280 −01	2.5148616 00	−9.2750921 00	3.4829391 01	−1.3542089 02	4.9165663 02
680	7332	−5.372621 −01	2.1957636 00	−7.7064045 00	2.7285914 01	−1.0001457 02	3.4272888 02
690	7238	−4.891831 −01	1.9157076 00	−6.4023513 00	2.1379568 01	−7.3888964 01	2.3903977 02
700	7141	−4.447081 −01	1.6698288 00	−5.3174144 00	1.6750843 01	−5.4591417 01	1.6675972 02
710	7042	−4.035787 −01	1.4539096 00	−4.4141926 00	1.3120790 01	−4.0325930 01	1.1632752 02
720	6940	−3.655575 −01	1.2642762 00	−3.6618564 00	1.0272379 01	−2.9774270 01	8.1115617 01
730	6834	−3.304273 −01	1.0977311 00	−3.0349583 00	8.0365194 00	−2.1966792 01	5.6520879 01
740	6726	−2.979876 −01	9.5148513 −01	−2.5124700 00	6.2812058 00	−1.6189359 01	3.9340504 01
750	6614	−2.680544 −01	8.2309892 −01	−2.0769630 00	4.9032127 00	−1.1914708 01	2.7341971 01
760	6499	−2.404576 −01	7.1044294 −01	−1.7139968 00	3.8217313 00	−8.7533620 00	1.8966972 01
770	6380	−2.150406 −01	6.1164807 −01	−1.4115692 00	2.9733723 00	−6.4170350 00	1.3126583 01
780	6258	−1.916595 −01	5.2507772 −01	−1.1597021 00	2.3083848 00	−4.6922170 00	9.0588082 00
790	6131	−1.701797 −01	4.4929375 −01	−9.5008533 −01	1.7876599 00	−3.4205830 00	6.2305662 00
800	6000	−1.504787 −01	3.8303207 −01	−7.7579091 −01	1.3804328 00	−2.4847057 00	4.2682900 00
810	5864	−1.324406 −01	3.2518071 −01	−6.3314586 −01	1.0744785 00	−1.7972981 00	
820	5724	−1.159602 −01	2.7476002 −01	−5.1494562 −01	8.1478628 −01	−1.2937330 00	
830	5578	−1.009390 −01	2.3090661 −01	−4.1696739 −01	6.2219116 −01	−9.2608262 −01	
840	5426	−8.72845 −02	1.9285776 −01	−3.3592570 −01	4.7288282 −01	−6.5857022 −01	
850	5268	−7.49130 −02	1.5994012 −01	−2.6906666 −01	3.5751464 −01	−4.6475168 −01	
860	5103	−6.37439 −02	1.3155841 −01	−2.1407952 −01	2.6871582 −01	−3.2503238 −01	
870	4931	−5.37042 −02	1.0718599 −01	−1.6902538 −01	2.0067934 −01	−2.2489016 −01	
880	4750	−4.47249 −02	8.6356386 −02	−1.3227511 −01	1.4882632 −01	−1.5360605 −01	
890	4560	−3.67420 −02	6.8656795 −02	−1.0246144 −01	1.0955479 −01	−1.0327183 −01	
900	4359	−2.96958 −02	5.3720383 −02	−7.8434618 −02	8.0029923 −02	−6.8063377 −02	
910	4146	−2.35297 −02	4.1222464 −02	−5.9228613 −02	5.8027193 −02	−4.3708815 −02	
920	3919	−1.81916 −02	3.0874445 −02	−4.4030118 −02	4.1882471 −02	−2.7088548 −02	
930	3676	−1.36330 −02	2.2420086 −02	−3.2159024 −02	2.9985729 −02	−1.5925460 −02	
940	3412	−9.8073 −03	1.5632010 −02	−2.3040871 −02	2.1516246 −02	−5.7833700 −03	
950	3122	−6.6715 −03	1.0307558 −02	−1.6194265 −02	1.5562408 −02	−8.5996330 −03	
960	2800	−4.1852 −03	6.2672193 −03	−1.1215535 −02	1.1481717 −02	−6.5585930 −03	
970	2431	−2.3107 −03	3.3509498 −03	−7.7662552 −03	8.7787174 −03	−8.7657040 −04	
980	1990	−1.0122 −03	1.4162907 −03	−5.5623627 −03	7.0714409 −03	−1.8324852 −04	
990	1411	−2.561 −04	3.3690093 −04	−4.3662509 −03	6.0693345 −03	−3.2675710 −05	

TABLE III. 1

Values of $Q^{(n)\prime}$, $n = 1, 2, 3, 4, 5, 6$ $(k = 1.405)$

T	M	$Q^{(1)\prime}$	$Q^{(2)\prime}$	$Q^{(3)\prime}$	$Q^{(4)\prime}$	$Q^{(5)\prime}$	$Q^{(6)\prime}$
400	9165	−3.3904604 02	1.1482342 04	−3.7528268 05	1.0868103 07	−3.0475886 08	8.1859411 09
401	9161	−3.3349352 02	1.1204621 04	−3.6323050 05	1.0434178 07	−2.9022584 08	7.7328348 09
402	9156	−3.2804208 02	1.0934154 04	−3.5158843 05	1.0018404 07	−2.7641313 08	7.3056602 09
403	9152	−3.2268980 02	1.0670735 04	−3.4034162 05	9.6199839 06	−2.6328382 08	6.9028830 09
404	9148	−3.1743460 02	1.0414167 04	−3.2947595 05	9.2381593 06	−2.5080258 08	6.5230623 09
405	9143	−3.1227456 02	1.0164259 04	−3.1897779 05	8.8722098 06	−2.3893643 08	6.1648501 09
406	9139	−3.0720772 02	9.9208224 03	−3.0883384 05	8.5214416 06	−2.2765369 08	5.8269716 09
407	9134	−3.0223217 02	9.6836778 03	−2.9903160 05	8.1851963 06	−2.1692458 08	5.5082366 09
408	9130	−2.9734618 02	9.4526471 03	−2.8955870 05	7.8628411 06	−2.0672084 08	5.2075202 09
409	9126	−2.9254787 02	9.2275621 03	−2.8040358 05	7.5537800 06	−1.9701581 08	4.9237740 09
410	9121	−2.8783551 02	9.0082590 03	−2.7155484 05	7.2574351 06	−1.8778412 08	4.6560058 09
411	9116	−2.8320738 02	8.7945742 03	−2.6300168 05	6.9732604 06	−1.7900181 08	4.4032871 09
412	9112	−2.7866184 02	8.5863584 03	−2.5473373 05	6.7007362 06	−1.7064625 08	4.1647481 09
413	9107	−2.7419717 02	8.3834537 03	−2.4674077 05	6.4393564 06	−1.6269574 08	3.9395592 09
414	9103	−2.6981181 02	8.1857180 03	−2.3901323 05	6.1886466 06	−1.5512998 08	3.7269525 09
415	9098	−2.6550426 02	7.9930112 03	−2.3154183 05	5.9481562 06	−1.4792972 08	3.5262039 09
416	9094	−2.6127286 02	7.8051915 03	−2.2431745 05	5.7174417 06	−1.3792494 08	3.3366269 09
417	9089	−2.5711609 02	7.6221231 03	−2.1733139 05	5.4960869 06	−1.3455277 08	3.1575766 09
418	9084	−2.5303263 02	7.4436836 03	−2.1057560 05	5.2837019 06	−1.2834245 08	2.9884564 09
419	9079	−2.4902092 02	7.2697414 03	−2.0404182 05	5.0799001 06	−1.2242728 08	2.8286919 09
420	9075	−2.4507968 02	7.1001798 03	−1.9772254 05	4.8843246 06	−1.1679980 08	2.6777533 09
421	9071	−2.4120737 02	6.9348736 03	−1.9161006 05	4.6966206 06	−1.1143861 08	2.5351321 09
422	9066	−2.3740272 02	6.7737099 03	−1.8569738 05	4.5164613 06	−1.0633286 08	2.4003572 09
423	9061	−2.3366450 02	6.6165805 03	−1.7997763 05	4.3435310 06	−1.0146919 08	2.2729844 09
424	9057	−2.2999132 02	6.4633729 03	−1.7444409 05	4.1775240 06	−9.6837948 07	2.1525924 09
425	9052	−2.2638192 02	6.3139816 03	−1.6909030 05	4.0181486 06	−9.2425291 07	2.0387849 09
426	9047	−2.2283505 02	6.1683037 03	−1.6391007 05	3.8651293 06	−8.8221284 07	1.9311904 09
427	9043	−2.1934963 02	6.0262445 03	−1.5889768 05	3.7182046 06	−9.3533356 07	1.8294622 09
428	9038	−2.1592433 02	5.8877018 03	−1.5404711 05	3.5771150 06	−8.0398920 07	1.7332654 09
429	9033	−2.1255807 02	5.7525840 03	−1.4935295 05	3.4416204 06	−7.6761518 07	1.6422915 09
430	9028	−2.0924974 02	5.6208017 03	−1.4480992 05	3.3114911 06	−7.3294894 07	1.5562488 09
431	9024	−2.0599816 02	5.4922630 03	−1.4041280 05	3.1865029 06	−6.9990636 07	1.4748599 09
432	9019	−2.0280232 02	5.3668847 03	−1.3615669 05	3.0664440 06	−6.6840921 07	1.3978655 09
433	9014	−1.9966108 02	5.2445805 03	−1.3203676 05	2.9511102 06	−6.3838197 07	1.3250202 09
434	9009	−1.9657353 02	5.1252741 03	−1.2804851 05	2.8403105 06	−6.0975423 07	1.2560949 09
435	9004	−1.9353850 02	5.0088804 03	−1.2418734 05	2.7338540 06	−5.8245745 07	1.1908697 09

436	8999	−1.9055506	02	4.8953262	03	−1.2044907	05	2.6315651	06	−5.5642788	07	1.1291407	09
437	8995	−1.8762223	02	4.7845357	03	−1.1682952	05	2.5332725	06	−5.3160419	07	1.0707135	09
438	8990	−1.8473911	02	4.6764387	03	−1.1332478	05	2.4388151	06	−5.0792903	07	1.0154077	09
439	8985	−1.8190471	02	4.5709629	03	−1.0993096	05	2.3480342	06	−4.8534685	07	9.6304986	08
440	8980	−1.7911815	02	4.4680418	03	−1.0664439	05	2.2607833	06	−4.6380576	07	9.1347875	08
441	8975	−1.7637848	02	4.3676062	03	−1.0346141	05	2.1769156	06	−4.4325538	07	8.6654003	08
442	8970	−1.7368485	02	4.2695923	03	−1.0037860	05	2.0962952	06	−4.2364857	07	8.2208993	08
443	8965	−1.7103642	02	4.1739391	03	−9.7392697	04	2.0187915	06	−4.0494090	07	7.7999290	08
444	8960	−1.6843239	02	4.0805860	03	−9.4500487	04	1.9442802	06	−3.8708952	07	7.4012100	08
445	8955	−1.6587181	02	3.9894693	03	−9.1698745	04	1.8726365	06	−3.7005318	07	7.0235074	08
446	8950	−1.6335397	02	3.9005355	03	−8.8984596	04	1.8037487	06	−3.5379410	07	6.6656993	08
447	8945	−1.6087806	02	3.8137266	03	−8.6355113	04	1.7375045	06	−3.3827292	07	6.3266910	08
448	8940	−1.5844329	02	3.7289884	03	−8.3807509	04	1.6737986	06	−3.2346106	07	6.0054724	08
449	8935	−1.5604892	02	3.6462674	03	−8.1339091	04	1.6125287	06	−3.0931915	07	5.7010736	08
450	8930	−1.5369418	02	3.5655128	03	−7.8947260	04	1.5535981	06	−2.9581763	07	5.4125907	08
451	8925	−1.5137840	02	3.4866741	03	−7.6629538	04	1.4969141	06	−2.8292665	07	5.1391671	08
452	8920	−1.4910082	02	3.4097028	03	−7.4383483	04	1.4423869	06	−2.7077613	07	4.8799885	08
453	8915	−1.4686071	02	3.3345506	03	−7.2206745	04	1.3899303	06	−2.5886267	07	4.6342899	08
454	8910	−1.4465748	02	3.2611728	03	−7.0097109	04	1.3394633	06	−2.4763676	07	4.4013754	08
455	8905	−1.4249039	02	3.1895225	03	−6.8052338	04	1.2909060	06	−2.3691480	07	4.1804839	08
456	8900	−1.4035881	02	3.1195576	03	−6.6070382	04	1.2441843	06	−2.2667349	07	3.9710513	08
457	8895	−1.3826207	02	3.0512348	03	−6.4149166	04	1.1992245	06	−2.1689043	07	3.7724331	08
458	8890	−1.3619955	02	2.9845125	03	−6.2286742	04	1.1559578	06	−2.0754444	07	3.5840591	08
459	8884	−1.3417063	02	2.9193510	03	−6.0481225	04	1.1143181	06	−1.9861527	07	3.4053841	08
460	8879	−1.3217472	02	2.8557111	03	−5.8730763	04	1.0742412	06	−1.9008373	07	3.2358920	08
461	8874	−1.3021125	02	2.7935554	03	−5.7033623	04	1.0356664	06	−1.8193149	07	3.0751002	08
462	8869	−1.2827958	02	2.7328458	03	−5.5388065	04	9.9853451	05	−1.7414103	07	2.9225443	08
463	8864	−1.2637915	02	2.6735451	03	−5.3792411	04	9.6278870	05	−1.6669560	07	2.7777884	08
464	8858	−1.2450945	02	2.6156208	03	−5.2245123	04	9.2837631	05	−1.5957962	07	2.6404252	08
465	8853	−1.2266986	02	2.5590366	03	−5.0744614	04	8.9524446	05	−1.5277782	07	2.5100636	08
466	8848	−1.2085989	02	2.5037603	03	−4.9289412	04	8.6334386	05	−1.4627593	07	2.3863361	08
467	8843	−1.1907902	02	2.4497596	03	−4.7878083	04	8.3262715	05	−1.4006029	07	2.2688954	08
468	8837	−1.1732670	02	2.3970022	03	−4.6509220	04	8.0304795	05	−1.3411778	07	2.1574109	08
469	8832	−1.1560248	02	2.3454589	03	−4.5181507	04	7.7456311	05	−1.2843608	07	2.0515743	08
470	8827	−1.1390580	02	2.2950976	03	−4.3893588	04	7.4712942	05	−1.2300318	07	1.9510867	08
471	8821	−1.1223620	02	2.2458908	03	−4.2644256	04	7.2070713	05	−1.1780796	07	1.8556715	08
472	8816	−1.1059318	02	2.1978091	03	−4.1432256	04	6.9525708	05	−1.1283955	07	1.7650652	08
473	8811	−1.0897630	02	2.1508256	03	−4.0256425	04	6.7074197	05	−1.0808772	07	1.6790167	08
474	8805	−1.0738509	02	2.1049131	03	−3.9115646	04	6.4712622	05	−1.0354274	07	1.5972910	08

TABLE III. 2

Values of $Q^{(n)\prime}$, $n = 1, 2, 3, 4, 5, 6$ $(k = 1.405)$

T	M	$Q^{(1)\prime}$	$Q^{(2)\prime}$	$Q^{(3)\prime}$	$Q^{(4)\prime}$	$Q^{(5)\prime}$	$Q^{(6)\prime}$
475	8800	−1.0581914 02	2.0600471 03	−3.8008832 04	6.2437607 05	−9.9195417 06	1.5196661 08
476	8794	−1.0427793 02	2.0161991 03	−3.6934858 04	6.0245706 05	−9.5036564 06	1.4459251 08
477	8789	−1.0276112 02	1.9733472 03	−3.5892768 04	5.8133896 05	−9.1058035 06	1.3758734 08
478	8784	−1.0126819 02	1.9314649 03	−3.4881509 04	5.6099017 05	−8.7251508 06	1.3093166 08
479	8778	−9.9798804 01	1.8905308 03	−3.3900168 04	5.4138246 05	−8.3609466 06	1.2460784 08
480	8773	−9.8352484 01	1.8505196 03	−3.2947767 04	5.2248675 05	−8.0124393 06	1.1859850 08
481	8767	−9.6928904 01	1.8114114 03	−3.2023463 04	5.0427715 05	−7.6789461 06	1.1288788 08
482	8762	−9.5527612 01	1.7731827 03	−3.1126335 04	4.8672693 05	−7.3597875 06	1.0746035 08
483	8756	−9.4148248 01	1.7358134 03	−3.0255577 04	4.6981175 05	−7.0543317 06	1.0230179 08
484	8751	−9.2790432 01	1.6992820 03	−2.9410353 04	4.5350729 05	−6.7619641 06	9.7398226 07
485	8745	−9.1453796 01	1.6635691 03	−2.8589889 04	4.3779092 05	−6.4821132 06	9.2736797 07
486	8740	−9.0137972 01	1.6286548 03	−2.7793417 04	4.2264036 05	−6.2142186 06	8.8305115 07
487	8734	−8.8842588 01	1.5945195 03	−2.7020191 04	4.0803447 05	−5.9577542 06	8.4091536 07
488	8728	−8.7567320 01	1.5611451 03	−2.6269509 04	3.9395278 05	−5.7122152 06	8.0085041 07
489	8723	−8.6311828 01	1.5285142 03	−2.5540692 04	3.8037614 05	−5.4771269 06	7.6275214 07
490	8717	−8.5075728 01	1.4966074 03	−2.4833039 04	3.6728505 05	−5.2520191 06	7.2651954 07
491	8711	−8.3858764 01	1.4654094 03	−2.4145938 04	3.5466208 05	−5.0364650 06	6.9206038 07
492	8706	−8.2660536 01	1.4349018 03	−2.3478737 04	3.4248933 05	−4.8300348 06	6.5928362 07
493	8700	−8.1480792 01	1.4050696 03	−2.2830849 04	3.3075056 05	−4.6323412 06	6.2810620 07
494	8695	−8.0319160 01	1.3758957 03	−2.2201668 04	3.1942909 05	−4.4429915 06	5.9844650 07
495	8689	−7.9175364 01	1.3473651 03	−2.1590642 04	3.0850991 05	−4.2616254 06	5.7022894 07
496	8683	−7.8049112 01	1.3194626 03	−2.0997216 04	2.9797813 05	−4.0878981 06	5.4338184 07
497	8678	−7.6940092 01	1.2921735 03	−2.0420857 04	2.8781939 05	−3.9214759 06	5.1783597 07
498	8672	−7.5848032 01	1.2654837 03	−1.9861054 04	2.7800999 05	−3.7620410 06	4.9352700 07
499	8666	−7.4772620 01	1.2393785 03	−1.9317304 04	2.6856674 05	−3.6092904 06	4.7039307 07
500	8660	−7.3713600 01	1.2138447 03	−1.8789126 04	2.5944686 05	−3.4629361 06	4.4837574 07
510	8602	−6.3966672 01	9.8714733 02	−1.4267723 04	1.8411336 05	−2.2959872 06	2.7870405 07
520	8542	−5.5586384 01	8.0480412 02	−1.0871187 04	1.3121808 05	−1.5302296 06	1.7444148 07
530	8480	−4.8365696 01	6.5768249 02	−8.3095980 03	9.3898456 04	−1.0248802 06	1.0992131 07
540	8417	−4.2131556 01	5.3863010 02	−6.3704926 03	6.7447771 04	−6.8957437 05	6.9723961 06
550	8352	−3.6738920 01	4.4202352 02	−4.8974308 03	4.8619799 04	−4.6596554 05	4.4516404 06
560	8285	−3.2065856 01	3.6342547 02	−3.7746896 03	3.5163421 04	−3.1613133 05	2.8608573 06
570	8216	−2.8009504 01	2.9932004 02	−2.9162739 03	2.5509332 04	−2.1527836 05	1.8507273 06
580	8146	−2.4482912 01	2.4691242 02	−2.2580206 03	1.8558244 04	−1.4710720 05	1.2053951 06
590	8074	−2.1412400 01	2.0397402 02	−1.7518660 03	1.3536534 04	−1.0084472 05	7.9062661 05
600	8000	−1.8735282 01	1.6872078 02	−1.3616560 03	9.8972235 03	−6.9333412 04	5.2242705 05
610	7924	−1.6398206 01	1.3972122 02	−1.0601067 03	7.2520257 03	−4.7795744 04	3.4793799 05

620	7846	−1.4355580 01	1.1582270 02	−8.2654985 02	5.3241493 03	−3.3027881 04	2.3370066 05
630	7766	−1.2568396 01	9.6094921 01	−6.4528074 02	3.9155276 03	−2.2872075 04	1.5841952 05
640	7684	−1.1003176 01	7.9784487 01	−5.0432385 02	2.8839418 03	−1.5869067 04	1.0846821 05
650	7599	−9.6311564 00	6.6280087 01	−3.9452330 02	2.1268699 03	−1.1028199 04	7.5080517 04
660	7513	−8.4275576 00	5.5084354 01	−3.0885757 02	1.5702074 03	−7.6748247 03	5.2587351 04
670	7424	−7.3709904 00	4.5791528 01	−2.4192689 02	1.1602058 03	−5.3465584 03	3.7305470 04
680	7332	−6.4429660 00	3.8069979 01	−1.8956873 02	8.5777335 02	−3.7278129 03	2.6825255 04
690	7238	−5.6274732 00	3.1648073 01	−1.4856626 02	6.3440357 02	−2.6005689 03	1.9565563 04
700	7141	−4.9106044 00	2.6302669 01	−1.1642714 02	4.6924975 02	−1.8146301 03	1.4481350 04
710	7042	−4.2802912 00	2.1850367 01	−9.1217254 01	3.4703600 02	−1.2661413 03	1.0878533 04
720	6940	−3.7260128 00	1.8139947 01	−7.1431653 01	2.5654198 02	−8.8310255 02	8.2931235 03
730	6834	−3.2386024 00	1.5046550 01	−5.5897644 01	1.8950799 02	−6.1549820 02	6.4130467 03
740	6726	−2.8100572 00	1.2466965 01	−4.3699609 01	1.3984606 02	−4.2852228 02	5.0269764 03
750	6614	−2.4333704 00	1.0315622 01	−3.4121447 01	1.0305738 02	−2.9790886 02	3.9906960 03
760	6499	−2.1024192 00	8.5215970 00	−2.6602482 01	7.5815929 01	−2.0671708 02	3.2050366 03
770	6380	−1.8118232 00	7.0259177 00	−2.0702828 01	5.5657476 01	−1.4310655 02	2.6011679 03
780	6258	−1.5568668 00	5.7795388 00	−1.6077184 01	4.0755201 01	−9.8790050 01	2.1308842 03
790	6131	−1.3334008 00	4.7415969 00	−1.2454062 01	2.9753032 01	−6.7968697 01	7.6007216 02
800	6000	−1.1377744 00	3.8780122 00	−9.6198553 00	2.1644134 01	−4.6577874 01	4.6432099 02
810	5864	−9.6677200 −01	3.1603308 00	−7.4063435 00	1.5680227 01		
820	5724	−8.1755680 −01	2.5647662 00	−5.6809946 00	1.1304888 01		
830	5578	−6.8762560 −01	2.0714246 00	−4.3393074 00	8.1047046 00		
840	5426	−5.7476400 −01	1.6636437 00	−3.2988895 00	5.7723933 00		
850	5268	−4.7701440 −01	1.3274667 00	−2.4947504 00	4.0797845 00		
860	5103	−3.9264360 −01	1.0511880 00	−1.8756271 00	2.8575629 00		
870	4931	−3.2011600 −01	8.2499092 −01	−1.4011235 00	1.9800465 00		
880	4750	−2.5806920 −01	6.4063276 −01	−1.0393788 00	1.3543241 00		
890	4560	−2.0529560 −01	4.9119807 −01	−7.6532359 −01			
900	4359	−1.6072024 −01	3.7087868 −01	−5.5922198 −01			
910	4146	−1.2339060 −01	2.7479368 −01	−4.0558238 −01			
920	3919	−9.2457880 −02	1.9884338 −01	−2.9224392 −01			
930	3676	−6.7167360 −02	1.3959274 −01	−2.0969192 −01			
940	3412	−4.6847960 −02	9.4151469 −02	−1.5049928 −01			
950	3122	−3.0900440 −02	6.0095524 −02	−1.0887718 −01			
960	2800	−1.8792920 −02	3.5394749 −02	−8.034294 −02			
970	2431	−1.0050200 −02	1.8347213 −02	−6.143819 −02			
980	1990	−4.2483600 −03	7.5289665 −03	−4.949538 −02			
990	1411	−1.0106800 −03	1.7489572 −03	−4.248428 −02			

TABLE IV. 0

The value of g [8]

θ \ $\lambda-\alpha$	.06	.04	.02	0	−.02	−.04
0	0 +.26630i	0 +.52445i	0 +.98607i		−1.41248	−1.13031
.05	+.44272 +.15738i	−.5275 +.26827i	−.65088 +.34045i	−.79871 +.33556i	−.89974 +.25757i	−.93314 +.17339i
.10	+.63903 +.05833i	−.66967 +.10353i	−.71308 +.12947i	−.75845 +.14013i	−.79942 +.13281i	−.82912 +.11632i
.15	+.72469 +.019106i	−.73600 +.03971i	−.75273 +.05517i	−.77070 +.06514i	−.78922 +.06870i	−.80572 +.06751i
.20	+.74868 +.004147i	−.76924 +.01448i	−.77702 +.02520i	−.78626 +.03265i	−.79533 +.03650i	−.80341 +.03798i
.25		−.79149 +.00521i	−.80035 +.01105i	−.79880 +.01593i	−.80356 +.01958i	−.80847 +.02211i
.30			−.80564 +.00417i	−.80789 +.00751i	−.81050 +.01045i	−.81336 +.01256i
.35				−.81462 +.00294i	−.81608 +.00507i	−.81774 +.00681i
.40					−.82042 +.00200i	−.82140 +.00335i

θ \ $\lambda-\alpha$	−.06	−.08	−.10	−.15	−.20
0	+1.02378	−.97101	−.93429	−.89106	−.87116
.05	+.92883 +.09723i	−.92906 +.07659i	−.90601 +.05286i	−.88168 +.0240i	−.86726 +.01306i
.10	+.84804 +.09671i	−.85836 +.07816i	−.86570 +.06795i	−.86269 +.03528i	−.85794 +.02095i
.15	+.81904 +.06306i	−.82932 +.05694i	−.83659 +.04989i	−.84593 +.03423i	−.84799 +.02304i
.20	+.81245 +.03890i	−.81948 +.03755i	−.82555 +.03534i	−.83544 +.02814i	−.84026 +.02114i
.25	+.81338 +.02365i	−.81785 +.02418i	−.82224 +.02382i	−.82956 +.02174i	−.83510 +.01758i
.30	+.81615 +.01408i	−.81912 +.01522i	−.82197 +.01573i	−.82792 +.01544i	−.83248 +.01379i
.35	+.81955 +.00811i	−.82131 +.00935i	−.82324 +.01009i	−.82752 +.01081i	−.83107 +.01043i
.40	+.82249 +.00453i	−.82372 +.00572i	−.82490 +.00630i	−.82797 +.00741i	−.83068 +.00770i
.45	+.82493 +.00213i	−.82568 +.00301i	−.82678 +.00350i	−.82871 +.00494i	−.83081 +.00566i
.50		−.82753 +.00132i	−.82811 +.00192i	−.82964 +.00311i	−.83120 +.00382i
.55				−.83058 +.00178i	−.83168 +.00249i
.60					−.83228 +.00152i
.65					−.83280 +.00075i

TABLE IV. 1

The value of g [1]

θ \ $\lambda-\alpha$	.06	.04	.02	0	−.02	−.04
0	0 +.06770i	0 +.0601i	0 +.04548i	0	−.0512	−.07640
.05	−.0113 +.07835i	−.0209 +.07404i	−.0329 +.06787i	−.0474 +.06104i	−.0645 +.05564i	−.08290 +.05144i
.10	−.0165 +.10695i	−.0296 +.10525i	−.0434 +.40298i	−.0581 +.10024i	−.0737 +.109746i	−.09002 +.09498i
.15	−.01844 +.14179i	−.0330 +.14111i	−.0478 +.14015i	−.0631 +.13894i	−.0787 +.13758i	−.09460 +.13621i
.20	−.101902+.18118i	−.0342 +.18035i	−.0497 +.17994i	−.0653 +.17936i	−.0811 +.17866i	−.09713 +.17790i
.25		−.03796 +.21668i	−.0504 +.22119i	−.0664 +.22092i	−.0825 +.22056i	−.09857 +.22014i
.30			−.0509 +.26190i	−.0670 +.26178i	−.0832 +.26160i	−.09941 +.26136i
.35				−.0672 +.30422i	−.0835 +.30410i	−.09986 +.30398i
.40					−.0836 +.34490i	−.10004 +.34485i

θ \ $\lambda-\alpha$	−.06	−.08	−.10	−.15	−.20
0	−.09858	−.11673	−.13710	−.18265	−.22712
.05	−.10244 +.04942i	−.12103 +.04744i	−.13875 +.04607i	−.18401 +.04433i	−.22745 +.04346i
.10	−.10739 +.09333i	−.12503 +.09160i	−.14150 +.09025i	−.18555 +.08801i	−.22836 +.08661i
.15	−.11149 +.13520i	−.12842 +.13381i	−.14417 +.13267i	−.18784 +.13073i	−.22947 +.12921i
.20	−.11397 +.17688i	−.13076 +.17554i	−.14628 +.17420i	−.18891 +.17275i	−.23059 +.17140i
.25	−.11552 +.21853i	−.13225 +.21719i	−.14773 +.21534i	−.19014 +.21426i	−.23155 +.21330i
.30	−.11644 +.26018i	−.13323 +.25892i	−.14870 +.25642i	−.19105 +.25581i	−.23233 +.25498i
.35	−.11699 +.30188i	−.13384 +.30065i	−.14933 +.29753i	−.19171 +.29721i	−.23293 +.29657i
.40	−.11730 +.34360i	−.13419 +.34239i	−.14974 +.33868i	−.19217 +.33860i	−.23340 +.33811i
.45	−.11755 +.38536i	−.13440 +.38412i	−.15000 +.38194i	−.19248 +.38001i	−.23373 +.37964i
.50		−.13450 +.42510i	−.15013 +.42330i	−.19267 +.42147i	−.23397 +.42119i
.55				−.19279 +.46297i	−.23412 +.46279
.60					−.23423 +.50435i
.65					−.23428 +.54598i

TABLE IV. 2

The value of g [2]

θ \ $\lambda-\alpha$	.06	.04	.02	0	−.02	−.04
0	0 +.00287i	0 +.00159i	0 +.00052i	0	+.0006	+.0019
.05	−.0035 +.00258i	−.0032 +.00102i	−.0028 −.00040i	−.0.0019 −0.00145	−.0009 −.00290i	+.0006 −.00399i
.10	−.0081 +.00186i	−.0077 −.00028i	−.0070 −.00233i	−.0.0060 −0.00421i	−.0047 −.00635i	−.0031 −.00832i
.15	−.01432 +.00104i	−.0138 −.00188i	−.0131 −.00460i	−.0.0119 −0.00725i	−.0104 −.01016i	−.0088 −.01295i
.20	−.02239 +.000103i	−.0218 −.00357i	−.0211 −.00705i	− 0.0198 −0.01045i	−.0183 −.01416i	−.0168 −.01769i
.25		−.0317 −.00537i	−.0311 −.00955i	− 0.0298 −0.01375i	−.0283 −.01825i	−.0266 −.02258i
.30			−.0432 −.01205i	− 0.0419 −0.01709i	−.0404 −.02239i	−.0387 −.02753i
.35				−.0.0560 −0.02045i	−.0545 −.02654i	−.0528 −.03215i
.40					−.0707 −.03069i	−.0690 −.03750i

θ \ $\lambda-\alpha$	−.06	−.08	−.10	−.15	−.20
0	+.0037	+.0058	+.0083	+.0163	+.0265
.05	+.0025 −.00501i	+.0046 −.00594i	+.0071 −.00689i	+.0152 −.00916i	+.0255 −.01137i
.10	−.0011 −.01026i	+.0011 −.01210i	+.0037 −.01389i	+.0119 −.01840i	+.0223 −.02276i
.15	−.0068 −.01574i	−.0045 −.01843i	−.0019 −.02103i	+.0064 −.02772i	+.0170 −.03420i
.20	−.0146 −.02138i	−.0122 −.02491i	−.0096 −.02829i	−.0012 −.03713i	+.0095 −.04570i
.25	−.0245 −.02712i	−.0220 −.03149i	−.0193 −.03564i	−.0109 −.04660i	−.0001 −.05725i
.30	−0.365 −.03292i	−.0340 −.03812i	−.0311 −.04306i	−.0226 −.05614i	−.0119 −.06885i
.35	−0.506 −.03876i	−.0479 −.04480i	−.0449 −.05051i	−.0364 −.06571i	−.0257 −.08048i
.40	−0.637 −.04462i	−.0640 −.05150i	−.0608 −.05799i	−.0523 −.07531i	−.0416 −.09214i
.45	−0.849 −.05049i	−.0822 −.05822i	−.0788 −.06548i	−.0703 −.08492i	−.0596 −.10382i
.50		−.1024 −.06494i	−.0989 −.07298i	−.0903 −.09455i	−.0796 −.11551i
.55				−.1125 −.10419i	−.1017 −.12731i
.60					−.1259 −.13902i
.65					−.1521 −.15073i

TABLE IV. 3

The values of $g^{[3]}$

θ \ $\lambda-\alpha$	.06	.04	.02	0	−.02	−.04
0	0 + .0092·10⁻⁶i	0 + .0251·10⁻⁶i	0 + .0047·10⁻⁶i	0	+ .0045·10⁻⁶	+ .027·10⁻⁶
.05	−.138·10⁻³ − .0108·10⁻⁶i	−.068·10⁻³ − .048·10⁻⁶i	−.007·10⁻³ − .053·10⁻⁶i	+ .035·10⁻³ − .038·10⁻⁶i	+ .059·10⁻³ + .0015·10⁻⁶i	+ .071·10⁻³ + .071·10⁻⁶i
.10	−.249·10⁻³ − .2958·10⁻⁶i	−.087·10⁻² − .315·10⁻⁶i	+.059·10⁻³ − .287·10⁻⁶i	+ .177·10⁻³ − .228·10⁻⁶i	+ .281·10⁻³ − .1300·10⁻⁶i	+ .377·10⁻³ + .020·10⁻⁶i
.15	−.323·10⁻³ − .8563·10⁻⁶i	−.034·10⁻² − .836·10⁻⁶i	+.231·10⁻³ − .771·10⁻⁶i	+ .461·10⁻³ − .667·10⁻⁶i	+ .694·10⁻³ − .488·10⁻⁶i	+ .900·10⁻³ − .271·10⁻⁶i
.20	−.351·10⁻³ −1.7741·10⁻⁶i	+.102·10⁻² −1.724·10⁻⁶i	+.521·10⁻³ −1.605·10⁻⁶i	+ .903·10⁻³ −1.449·10⁻⁶i	+1.302·10⁻³ −1.198·10⁻⁶i	+1.675·10⁻³ − .896·10⁻⁶i
.25		+.326·10⁻² −3.002·10⁻⁶i	+.926·10⁻³ −2.909·10⁻⁶i	+1.508·10⁻³ −2.667·10⁻⁶i	+2.112·10⁻³ −2.368·10⁻⁶i	+2.682·10⁻³ −1.978·10⁻⁶i
.30			+1.476·10⁻³ −4.766·10⁻⁶i	+2.279·10⁻³ −4.461·10⁻⁶i	+3.128·10⁻³ −4.078·10⁻⁶i	+3.935·10⁻³ −3.608·10⁻⁶i
.35				+3.217·10⁻³ −6.908·10⁻⁶i	+4.351·10⁻³ −6.446·10⁻⁶i	+5.436·10⁻³ −5.993·10⁻⁶i
.40					+5.782·10⁻³ −9.546·10⁻⁶i	+7.186·10⁻³ −8.938·10⁻⁶i

θ \ $\lambda-\alpha$	−.06	−.08	−.10	−.15	−.20
0	+.0795·10⁻³	+.1735·10⁻³	+.3130·10⁻³	+.9435·10⁻³	+2.0110·10⁻³
.05	+.046·10⁻³ +.164·10⁻⁶i	+.0276·10⁻³ +.313·10⁻⁶i	+.125·10⁻³ +.375·10⁻⁶i	+.7145·10⁻³ + .787·10⁻⁶i	+1.7268·10⁻³ +1.287·10⁻⁶i
.10	+.428·10⁻³ +.210·10⁻⁶i	+.3895·10⁻³ +.479·10⁻⁶i	+.354·10⁻³ +.646·10⁻⁶i	+.0255·10⁻³ +1.475·10⁻⁶i	+.8736·10⁻³ +2.425·10⁻⁶i
.15	+1.078·10⁻³ +.022·10⁻⁶i	+1.275·10⁻³ +.417·10⁻⁶i	+1.313·10⁻³ +.688·10⁻⁶i	+2.748·10⁻³ +1.950·10⁻⁶i	+.5003·10⁻³ +3.337·10⁻⁶i
.20	+2.006·10⁻³ −.473·10⁻⁶i	+2.359·10⁻³ +.021·10⁻⁶i	+2.523·10⁻³ +.417·10⁻⁶i	+4.841·10⁻³ +1.825·10⁻⁶i	+3.5479·10⁻³ +3.969·10⁻⁶i
.25	+3.219·10⁻³ −1.418·10⁻⁶i	+3.774·10⁻³ −.813·10⁻⁶i	+4.066·10⁻³ −.313·10⁻⁶i	+7.409·10⁻³ +1.037·10⁻⁶i	+5.1217·10⁻³ +4.219·10⁻⁶i
.30	+4.720·10⁻³ −2.938·10⁻⁶i	+5.466·10⁻³ −2.168·10⁻⁶i	+6.192·10⁻³ −1.543·10⁻⁶i	+10.455·10⁻³ −.437·10⁻⁶i	+8.2742·10⁻³ +3.956120·10⁻⁶i
.35	+6.512·10⁻³ −5.103·10⁻⁶i	+7.489·10⁻³ −4.191·10⁻⁶i	+8.444·10⁻³ −3.440·10⁻⁶i	+13.981·10⁻³ −2.637·10⁻⁶i	+12.0074·10⁻³ +3.031·10⁻⁶i
.40	+8.599·10⁻³ −8.028·10⁻⁶i	+9.908·10⁻³ −5.985·10⁻⁶i	+11.363·10⁻³ −6.026·10⁻⁶i	+17.986·10⁻³ −5.650·10⁻⁶i	+16.3229·10⁻³ +1.344·10⁻⁶i
.45	+10.973·10⁻³ −11.813·10⁻⁶i	+13.598·10⁻³ −10.675·10⁻⁶i	+14.324·10⁻³ −9.487·10⁻⁶i	+22.472·10⁻³ −9.650·10⁻⁶i	+21.2219·10⁻³ +1.206·10⁻⁶i
.50		+18.143·10⁻³ −15.221·10⁻⁶i	+18.661·10⁻³ −13.928·10⁻⁶i	+27.441·10⁻³ −14.713·10⁻⁶i	+26.7051·10⁻³ +4.706·10⁻⁶i
.55					+32.7756·10⁻³ +9.206·10⁻⁶i
60					+39.4338·10⁻³ +14.833·10⁻⁶i
.65					+46.6776·10⁻³ +21.694·10⁻⁶i

TABLE IV. 4
The values of $g^{[4]}$

θ \ $\lambda-\alpha$	.06	.04	.02	0	−.02	−.04
0	0 +.0113·10⁻⁴i	0 +.00288·10⁻⁴i	0 +.036·10⁻⁴i	0	+.000225·10⁻⁴	+.0029·10⁻⁴
.05	−.0184·10⁻⁴ −.0242·10⁻⁴i	+.0008·10⁻⁴ −.014 ·10⁻⁴i	+.012·10⁻⁴ −.003·10⁻⁴i	+.008·10⁻⁴ +.006·10⁻⁴i	−.00065 ·10⁻⁴ +.0125 ·10⁻⁴i	−.017 ·10⁻⁴ +.00525·10⁻⁴i
.10	+.0578·10⁻⁴ −.1212·10⁻⁴i	+.057 ·10⁻⁴ −.056 ·10⁻⁴i	+.090·10⁻⁴ +.008·10⁻⁴i	+.064·10⁻⁴ +.054·10⁻⁴i	+.0304 ·10⁻⁴ +.10625·10⁻⁴i	−.045 ·10⁻⁴ +.11875·10⁻⁴i
.15	+.3468·10⁻⁴ −.2640·10⁻⁴i	+.371 ·10⁻⁴ −.089 ·10⁻⁴i	+.342·10⁻⁴ +.078·10⁻⁴i	+.280·10⁻⁴ +.207·10⁻⁴i	+.1734 ·10⁻⁴ +.35625·10⁻⁴i	+.0075·10⁻⁴ +.46875·10⁻⁴i
.20	+.9994·10⁻⁴ −.4323·10⁻⁴i	+1.007 ·10⁻⁴ −.074 ·10⁻⁴i	+.924·10⁻⁴ +.361·10⁻⁴i	+.796·10⁻⁴ +.541·10⁻⁴i	+.4689 ·10⁻⁴ +.85625·10⁻⁴i	+.265 ·10⁻⁴ +1.14375·10⁻⁴i
.25		+2.204 ·10⁻⁴ −.181 ·10⁻⁴i	+2.036·10⁻⁴ +.821·10⁻⁴i	+1.812·10⁻⁴ +1.136·10⁻⁴i	+1.324 ·10⁻⁴ +1.66125·10⁻⁴i	+.965 ·10⁻⁴ +2.25625·10⁻⁴i
.30			+3.952·10⁻⁴ +1.222·10⁻⁴i	+3.584·10⁻⁴ +2.068·10⁻⁴i	+2.919 ·10⁻⁴ +2.9625 ·10⁻⁴i	+2.375 ·10⁻⁴ +3.90625·10⁻⁴i
.35				+6.412·10⁻⁴ +3.410·10⁻⁴i	+5.494 ·10⁻⁴ +3.410 ·10⁻⁴i	+4.735 ·10⁻⁴ +6.21875·10⁻⁴i
.40					+9.449 ·10⁻⁴ +7.3 ·10⁻⁴i	+8.385 ·10⁻⁴ +9.38125·10⁻⁴i

θ \ $\lambda-\alpha$	−.06	−.08	−.10	−.15	−.20
0	+.01318·10⁻⁴	+.03698·10⁻⁴	+.0460·10⁻⁴	+.3175·10⁻⁴	+1 .028·10⁻⁴
.05	−.028 ·10⁻⁴ −.125 ·10⁻⁴i	−.045 ·10⁻⁴ −.0375·10⁻⁴i	−.0520·10⁻⁴ −.1125·10⁻⁴i	+.0675·10⁻⁴ −.45 ·10⁻⁴i	+.653·10⁻⁴ −.950·10⁻⁴i
.10	−.125 ·10⁻⁴ +.1000·10⁻⁴i	+.252 ·10⁻⁴ +.0625·10⁻⁴i	+.3130·10⁻⁴ −.0500·10⁻⁴i	−.5675·10⁻⁴ −.65 ·10⁻⁴i	−.347·10⁻⁴ −1.60 ·10⁻⁴i
.15	−.304 ·10⁻⁴ +.4625·10⁻⁴i	+.492 ·10⁻⁴ +.45 ·10⁻⁴i	+.657 ·10⁻⁴ +.3500·10⁻⁴i	+1.5075·10⁻⁴ −.40 ·10⁻⁴i	−1.797·10⁻⁴ −1.70 ·10⁻⁴i
.20	−.066 ·10⁻⁴ +1.2125·10⁻⁴i	+.615 ·10⁻⁴ +1.3625·10⁻⁴i	+.944 ·10⁻⁴ +1.275 ·10⁻⁴i	+2.5075·10⁻⁴ −4.25 ·10⁻⁴i	−3.622·10⁻⁴ −.800·10⁻⁴i
.25	+.357 ·10⁻⁴ +2.4675·10⁻⁴i	+.442 ·10⁻⁴ +2.925 ·10⁻⁴i	+1.008 ·10⁻⁴ +2.8875·10⁻⁴i	+3.2325·10⁻⁴ +2.175·10⁻⁴i	−5.722·10⁻⁴ +1.20 ·10⁻⁴i
.30	−1.387 ·10⁻⁴ +4.425 ·10⁻⁴i	+.302 ·10⁻⁴ +5.275 ·10⁻⁴i	+.558 ·10⁻⁴ +5.3875·10⁻⁴i	+3.9825·10⁻⁴ +5.20 ·10⁻⁴i	−7.822·10⁻⁴ +4.55 ·10⁻⁴i
.35	+3.337 ·10⁻⁴ +7.1675·10⁻⁴i	+1.894 ·10⁻⁴ +8.525 ·10⁻⁴i	+.622 ·10⁻⁴ +9.0000·10⁻⁴i	+4.1825·10⁻⁴ +9.65 ·10⁻⁴i	−9.507·10⁻⁴ +9.60 ·10⁻⁴i
.40	+6.612 ·10⁻⁴ +10.9375·10⁻⁴i	+4.657 ·10⁻⁴ +11.2637·10⁻⁴i	+2.962 ·10⁻⁴ +13.9500·10⁻⁴i	+3.4825·10⁻⁴ +15.775·10⁻⁴i	+10.672·10⁻⁴ +16.7 ·10⁻⁴i
.45	+11.562 ·10⁻⁴ +15.875 ·10⁻⁴i	+9.037 ·10⁻⁴ +18.625 ·10⁻⁴i	+6.852 ·10⁻⁴ +20.275 ·10⁻⁴i	+1.5325·10⁻⁴ +23.775·10⁻⁴i	+10.722·10⁻⁴ +26.1 ·10⁻⁴i
.50		+15.499 ·10⁻⁴ +26.5125·10⁻⁴i	+12.453 ·10⁻⁴ +28.175 ·10⁻⁴i	+2.2925·10⁻⁴ +33.85 ·10⁻⁴i	+9.297·10⁻⁴ +38.1 ·10⁻⁴i
.55				+8.3675·10⁻⁴ +46.45 ·10⁻⁴i	+5.773·10⁻⁴ +53.2 ·10⁻⁴i
.60					+.278·10⁻⁴ +71.5 ·10⁻⁴i
.65					+9.353·10⁻⁴ +92.6 ·10⁻⁴i

§3), the $\psi_\nu(\lambda, \theta)$ are a set of solutions of equation (3.5), and $X_\nu(\lambda, \theta)$ and $Y_\nu(\lambda, \theta)$ are certain functions, given by (3.7), which are connected with the $\psi_\nu(\lambda, \theta)$.

The first question which arises is that of deciding which set of functions

TABLE V

The values of H (2λ) and $Q^{(n)}$ (2λ) for values of λ needed for the example under consideration

$\lambda-\alpha$	T	M	H	$Q^{(1)}$	$Q^{(2)}$	$Q^{(3)}$	$Q^{(4)}$
.06	.758	.652	1.0372	−.2460	.7330	−1.6551	4.0380
.04	.774	.633	1.0323	−.2057	.5770	−1.2009	2.7074
.02	.787	.617	1.0285	−.1766	.4720	−.9246	1.9439
0	.800	.600	1.0250	−.1505	.3830	−.6954	1.3804
−.02	.811	.585	1.0222	−.1308	.3201	−.5514	1.0485
−.04	.822	.569	1.0197	−.1130	.2660	−.4350	.7763
−.06	.832	.555	1.0175	−.0982	.2233	−.3483	.5923
−.08	.841	.541	1.0156	−.0860	.1896	−.2833	.4614
−.10	.850	.527	1.0139	−.0749	.1599	−.2291	.3575
−.15	.868	.497	1.0107	−.0557	.1121	−.1482	.2143
−.20	.884	.467	1.0083	−.0415	.0793	−.0981	.1331

TABLE VI

The values of the streamfunction in the λ, θ plane

θ \ λ−α	.06	.04	.02	0	−.02	−.04	−.06	−.08	−.10	−.15	−.20
0	.269	.536	1.010		0	0	0	0	0	0	0
.05	.155	.270	.344	.339	.259	.173	.095	.075	.031	.022	.011
.10	.049	.096	.124	.135	.128	.112	.092	.073	.064	.031	.017
.15	−.004	.027	.044	.055	.059	.059	.055	.049	.043	.028	.018
.20	−.014	−.002	.010	.019	.024	.026	.028	.027	.026	.020	.014
.25		−.013	−.006	0	.005	.008	.011	.012	.013	.009	.012
.30			−.014	−.010	−.006	−.003	−.001	.001	.003	.004	.004
.35				−.014	−.012	−.009	−.007	−.005	−.004	−.002	0
.40					−.013	−.012	−.011	−.009	−.008	−.006	−.004
.45							−.012	−.011	−.010	−.008	−.006

$\psi_\nu(\lambda, \theta)$ should be chosen in order that (2.1) may yield a large variety of flow patterns which are of interest.[4]

As a starting point, let us consider the flows of an *incompressible* fluid. Then

[4] The question of determining which set of solutions $\psi_\nu(\lambda, \theta)$ should be chosen in order that (2.1) may yield *all* possible flow patterns of a certain type represents a very interesting mathematical topic which will be discussed elsewhere.

a rather large class of symmetric flows around a closed obstacle can be obtained (see §3) by choosing for the set $\{\psi_\nu^\dagger\}$ the real and imaginary parts of the functions:

$$(2.2)\quad \psi_{2\kappa}^\dagger(\lambda^\dagger, \theta) + i\psi_{2\kappa}^\dagger(\lambda^\dagger, \theta) = (\exp Z^\dagger - \exp\alpha)^{\kappa/2}, \kappa = -1, 0, 1, 2, \cdots$$

$$Z^\dagger = \lambda^\dagger - i\theta, \qquad \lambda^\dagger = \log q, \qquad \alpha = \text{arbitrary constant}, \qquad \exp Z^\dagger \equiv e^{Z^\dagger}$$

Here $q \exp(i\theta)$ is the velocity vector, q being the speed and θ the angle which the velocity vector makes with the positive direction of the x-axis.

TABLE VII. 1

The streamline $\psi = 0$ (obstacle) in the physical plane of the compressible flow pattern

q	θ	ρ	T^3	ψ_θ	ψ_q	x	y
.4571	.348	.899	.782	−.077	−.036	0	0
.4850	.335	.886	.753	−.106	−.053	.0104	.00354
.5127	.321	.874	.722	−.132	−.082	.0216	.00750
.5256	.312	.867	.707	−.152	−.119	.0278	.00956
.5384	.302	.860	.692	−.173	−.161	.0351	.00189
.5513	.288	.855	.676	−.194	−.215	.0439	.01456
.5657	.273	.848	.658	−.222	−.295	.0561	.01805
.5792	.255	.841	.640	−.264	−.449	.0724	.02242
.5945	.229	.832	.619	−.314	−.691	.1011	.02927

TABLE VII. 2

The streamline $\psi = .01$ in the physical plane of the compressible flow pattern

q	θ	ρ	T^3	ψ_θ	ψ_q	x	y
.4571	.237	.899	.782	−.100	.239	−.0942	.0679
.4850	.263	.886	.753	−.154	.144	−.0702	.0742
.5127	.268	.874	.722	−.200	.036	−.0505	.0796
.5256	.261	.867	.707	−.228	−.124	−.0418	.0820
.5384	.254	.860	.692	−.275	−.150	−.0319	.0846
.5513	.246	.855	.676	−.287	−.178	−.0198	.0874
.5657	.235	.848	.658	−.305	−.233	−.0029	.0905
.5792	.223	.841	.640	−.396	−.352	.0196	.0939
.5945	.202	.832	.619	−.486	−.667	.0479	.0993
.6088	.177	.825	.599	−.544	−.864	.0814	.1058
.6256	.132	.816	.575	−.708	−1.967	.1204	.1184

In analogy to the above set we may choose in the *compressible* case the class of functions $\psi_\nu(\lambda, \theta)$ given by[5]:

$$(2.3)\quad \psi_{2\kappa}(\lambda, \theta) + i\psi_{2\kappa+1}(\lambda, \theta) = P[(\exp Z - \exp\alpha)^{\kappa/2}], \kappa = -1, 0, 1, 2, \cdots$$

[5] Let us note that $\exp\alpha$ is the speed at infinity. A closer mathematical investigation shows that in the $\lambda^\dagger$, θ-plane the complex potential of the flows to which we refer have a branchpoint at the point $\lambda^\dagger = \alpha$, which point corresponds to infinity in the physical plane.

The integral operator to which we refer in the present paper is an operator which generates from analytic functions of a complex variable (complex) solutions of equation (3.5), where ψ is considered as a function of λ and θ. Here, as in the incompressible case, θ signifies the angle which the velocity vector makes with some fixed direction, while λ is a certain function of the speed (corresponding to $\lambda^\dagger = \lg q$ in the incompressible case), see (3.4).

where P is the "integral operator of the second kind" (see [1] – [5]), which generates solutions of the compressibility equation having many properties similar to those of the analytic functions to which the operator is applied.

For each fixed value of the parameter α it is possible, by applying the operator P, to generate a set (2.3) of solutions of the compressibility equation. From this set of solutions it is then possible to determine, by the use of (3.7), the corresponding functions $X_\nu(\lambda, \theta)$ and $Y_\nu(\lambda, \theta)$, so that by the use of (2.1) it would be possible to construct a broad class of compressible flow patterns. A slight modification of this procedure, which may be more suitable in practical applications, is the following:

Suppose that we are interested in determining flows past certain types of profiles, say Joukowski profiles or those of elliptical shape. By solving the corresponding *incompressible* flow problems, determining the complex potentials in the logarithmic plane ($Z^\dagger = \lambda^\dagger - i\theta$), and applying the P-operator to these functions, (with $\lambda^\dagger$ replaced by λ) there are obtained complex solutions of the compressibility equation whose imaginary parts give the streamfunctions (in the $Z = \lambda - i\theta$ plane) of a compressible fluid flow past an obstacle bearing a certain resemblance to that appearing in the corresponding incompressible flow. By adding to this streamfunction, which we may designate as a *"basic solution,"* the streamfunctions $\psi_{2\kappa}$, $\psi_{2\kappa+1}$ corresponding to[6] $\kappa = 0, 1, 2, \cdots$ in (2.3) one can, by taking various choices of the coefficients A_ν in (2.1), construct a class of compressible fluid flows which may be looked upon as modifications of that given by the "basic solution".

It may be noted that for even values of κ the functions to which the P-operator is applied become polynomials in exp Z, so that it suffices to determine the functions $P(\exp nZ)$, $n = 0, 1, 2, \cdots$. For *odd* values of κ this simplification is not possible.

Thus we obtain three sets of solutions:

A. Basic solutions.

B. Functions obtained by applying the P-operator to integral powers of $(\exp Z - \exp \alpha)^{\kappa/2}$, $\kappa = 1, 3, 5, 7, \cdots$ and for α assuming a sufficiently dense (discrete) set of values. (A *two* parameter family of functions.)

C. Functions obtained from exp (nZ) in the same manner as described in B. (A *one* parameter family of functions.)

In §3 and §4 we shall describe in more detail how to determine a basic solution, namely a flow pattern past an obstacle of oval shape.

3. Hodograph method in the case of incompressible and compressible fluid. Integral operators, generating streamfunctions of flows of a compressible fluid. In this section we wish to sketch the procedure used in the hodograph method in both the incompressible and compressible cases.

[6] The case $\kappa = -1$ is omitted, for the "basic solution" will contain such a singularity. Cf. §4.

We note that in order to avoid the necessity of adding to many terms of solutions listed under B and C, it is useful to prepare not one basic flow pattern, but for each type of obstacle a number of "basic solutions" corresponding to different values of any parameter, e.g. for an ellipse for a number of values of the eccentricity.

In the case of a steady two-dimensional flow, at every point (x, y) of the flow the velocity vector $\mathfrak{q} = qe^{i\theta}$ is defined. Here q is the speed and θ the angle which the velocity vector makes with some fixed direction, say the positive direction of the x-axis.

In the hodograph method one considers the potential (φ) and the streamfunction (ψ) as functions of q and θ, or, what is for some purposes more convenient (in the incompressible fluid case), as functions of[7] $\lambda' = \lg q$ and θ. ψ is (in the incompressible case) a harmonic function of λ' and θ; i.e., it satisfies the equation

$$\partial^2\psi/\partial\theta^2 + \partial^2\psi/\partial\lambda'^2 = 0 \tag{3.1}$$

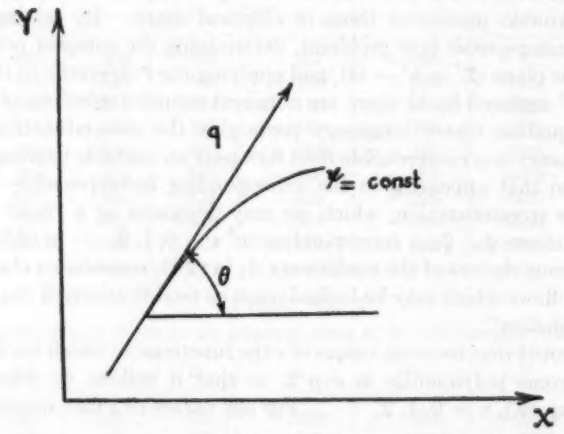

FIG. 1. A streamline in the physical plane, and the velocity vector

Taking the imaginary (or real) part of

$$g(Z'), \qquad Z' = \lambda' - i\theta \tag{3.2}$$

where g is any analytic function of the complex variable Z', one obtains a possible[8] streamfunction of an incompressible fluid flow (defined in the logarithmic plane).

In the incompressible fluid case it is not difficult to determine functions g which yield (in the physical plane) flows around profiles of certain types. For example, the function $g(Z)$ corresponding to the flow past an ellipse is given in §4.

As we mentioned in the introduction, the idea sketched above can be gener-

[7] The plane whose cartesian coordinates are λ' and θ is often denoted as the *logarithmic plane*.

[8] In general, the imaginary part of $g(Z)$ may yield, upon transition to the physical plane, a flow in which the velocity distribution is not single-valued. Such flows, of course, are physically meaningless. For this reason, we speak of *possible* streamfunctions. However, it should be remarked that by considering the streamfunction in only a *part* of its domain of definition we may obtain physically acceptable flows.

alized to the case of a compressible fluid. We assume that the flow is adiabatic, the pressure p and density ρ being related by the equation

$$p = A + \sigma\rho^k \tag{3.3}$$

where A, k and σ are constants. In the case of *subsonic* flows one has to work in the so-called *pseudo-logarithmic plane*, instead of in the logarithmic plane, i.e., the plane whose cartesian coordinates are

$$\lambda = \tfrac{1}{2}\left[\lg\left(\frac{1-T}{1+T}\right) - \frac{1}{h}\lg\left(\frac{1+hT}{1-hT}\right)\right], \tag{3.4}$$

and θ, where $h = \left(\frac{k-1}{k+1}\right)^{\frac{1}{2}}$, $T = (1-M^2)^{\frac{1}{2}}$ and M, the so-called Mach number, is equal to $\frac{q}{a_0}\left[1 - \tfrac{1}{2}(k-1)\frac{q^2}{a_0^2}\right]^{-\frac{1}{2}}$, k being the constant appearing in (3.3) and $a_0 = [(dp/d\rho)^{\frac{1}{2}}]_{q=0}$ the velocity of sound at a stagnation point. (Note that as M varies between 0 and 1, λ ranges between $-\infty$ and 0.)

In table I in the first three columns the corresponding values of T, M and λ are listed, for $k = 1.405$. In the following the functions $H(2\lambda) = (1-M^2)^{-1/4}\,[1 + \tfrac{1}{2}(k-1)M^2]^{-1/2(k-1)}$ and $\frac{dH(2\lambda)}{d\lambda}$ are needed. In columns 4 and 5 of Table I these values are given.

In the pseudo-logarithmic plane the streamfunction ψ satisfies the equation:

$$\frac{\partial^2\psi}{\partial\lambda^2} + \frac{\partial^2\psi}{\partial\theta^2} + 4N\frac{\partial\psi}{\partial\lambda} = 0, \qquad N = -\frac{k+1}{8}\cdot\frac{M^4}{(1-M^2)^{3/2}} \tag{3.5}$$

(Since λ is a function of M alone the quantity N may be considered as a known, although rather complicated, function of λ.)

While equation (3.5) is much simpler than the non-linear one which holds in the physical plane, its investigation still constitutes a considerable task. The theory of integral operators has been very useful in the study of (3.5). By employing the method of integral operators it is possible to transform any analytic function $g(Z)$ into a complex solution of (3.5), so that the imaginary (or real) part of the latter is a possible streamfunction of a compressible fluid flow. Thus, the method of integral operators makes it possible to generalize to compressible flows the much simpler operation, applicable in the case of incompressible flows, of taking the imaginary (or real) part of an analytic function.

The application of the method of integral operators to the problem of determining solutions of (3.5) may be described briefly as follows: There exist a set of functions $Q^{(n)}(2\lambda)$, $n = 1, 2, \cdots$ such that the series:

$$\mathrm{P}[g^{[0]}] \equiv (1-M^2)^{-\frac{1}{4}}[1 + \tfrac{1}{2}(k-1)M^2]^{-(1/2)(k-1)} \times \left\{g^{[0]}(Z) + \sum_{n=1}^{\infty}\frac{(2n)!}{2^{2n}n!}Q^{(n)}(2\lambda)g^{[n]}(Z)\right\}, \tag{3.6}$$

converges in a certain portion of the half-plane $\lambda < 0$ and constitutes a (complex) solution of (3.5), so that the imaginary part of $\mathrm{P}(g)$ is a possible streamfunction

of a compressible flow. In (3.6) $g^{[0]}(Z)$ is any analytic function of $Z = \lambda - i\theta$ and $g^{[n]}(Z) = \int_{\alpha}^{Z} g^{[n-1]}(Z')dZ'$, where α is any fixed value of Z, $\operatorname{Re}\alpha \leqq 0$. (The domain of convergence depends on the choice of α. In particular, if $\alpha = 0$, then the series converges in the interior of the wedge-shaped region $|\theta| \leqq 3^{\frac{1}{2}}|\lambda|$.)

In tables II.1 and II.2 the values of the $Q^{(n)}$'s for $n = 1, 2, 3, 4, 5, 6$ are listed. Since in further applications $Q^{(n)\prime} = dQ^{(n)}/d\lambda$ are needed, the values of this function are given in tables III.1 and III.2.[9] We use in these tables T as argument. Since the values of the $Q^{(n)}$ and $Q^{(n)\prime}$ are very large, each value is represented by two numbers, the number in the second column indicating the power of 10 by which the number in the first column is to be multiplied in order to obtain the value of the corresponding $Q^{(n)}$ or $Q^{(n)\prime}$. For instance

$$1.32456\ 01 \text{ means } 1.32456 \times 10 = 13.2456.$$

The values of M and λ as functions of T are given in Table I. In particular, for small values of M the streamfunction ψ reduces essentially to the imaginary part of $g(Z)$, so that the operator $P(g)$ may be considered as a transformation of an incompressible flow into a corresponding compressible flow.

In general, a part of the flow will be outside the domain of convergence of the series (3.6), so that it is necessary to extend the region of definition of ψ. This question will be discussed briefly in §5.

After a streamfunction has been determined in the λ, θ-plane by means of the operator (3.6), the task still remains of determining the flow in the physical plane—i.e., to express ψ as a function of x and y. The transition is almost immediate, for the mapping of the λ, θ-plane upon the x, y-plane is given by the following formulas:

$$(3.7)\quad \begin{cases} x = \int \frac{1}{\rho}\left\{\left[-\frac{(1-M^2)^{\frac{1}{2}}\cos\theta}{q}\psi_\theta - \frac{\sin\theta}{q}\psi_\lambda\right]d\lambda \right. \\ \qquad\qquad \left. + \left[\frac{(1-M^2)^{\frac{1}{2}}\cos\theta}{q}\psi_\lambda - \frac{\sin\theta}{q}\psi_\theta\right]d\theta\right\} \\ y = \int \frac{1}{\rho}\left\{\left[-\frac{(1-M^2)^{\frac{1}{2}}\sin\theta}{q}\psi_\theta + \frac{\cos\theta}{q}\psi_\lambda\right]d\lambda \right. \\ \qquad\qquad \left. + \left[\frac{(1-M^2)^{\frac{1}{2}}\sin\theta}{q}\psi_\lambda + \frac{\cos\theta}{q}\psi_\theta\right]d\theta\right\} \end{cases}$$

4. Illustrative Example of the Determination of a Compressible Fluid Flow by the Hodograph Method. Let us first consider the flow of an *incompressible* fluid (whose density we may take equal to one). As is well known, the use of analytic functions is very convenient in the study of such flows.

Let there be given a circle $|z| = R$, and suppose that it is desired to deter-

[9] The authors take this opportunity of expressing their gratitude to Miss H. L. Basset of the Center of Analysis, Massachusetts Institute of Technology, for supervising the computation of the $Q^{(n)}$ and $Q^{(n)\prime}$.

mine the (symmetric) flow past this obstacle having velocity $\frac{1}{2}U$ at infinity, where U is a positive real number. Then it is easily proven that the so-called "complex potential" $w = \phi + i\psi$ of the flow is given by:

$$w = \phi + i\psi = \tfrac{1}{2}U(z + R^2/z) \tag{4.1}$$

where, as usual, ϕ and ψ are the potential and streamfunctions respectively, and dw/dz is the conjugate of the velocity vector (*not* the velocity vector itself). See Fig. 2.

Now consider the mapping:

$$\zeta = \tfrac{1}{2}(z + z^{-1}). \tag{4.2}$$

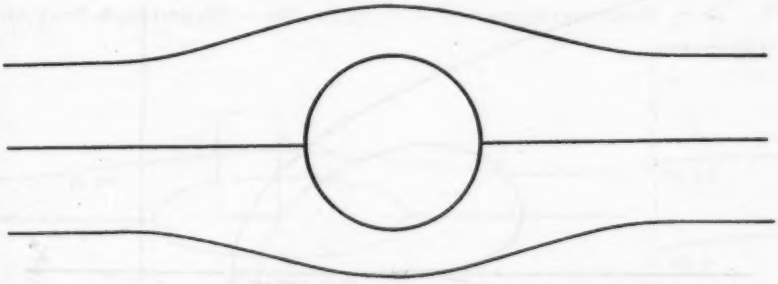

FIG. 2. An incompressible fluid flow pattern past a circle in the physical plane

FIG. 3. An incompressible fluid flow pattern past an ellipse in the physical plane

If, as we shall assume, $R > 1$, this mapping transforms the exterior of the aforementioned circle into the exterior of the ellipse indicated in Fig. 3.
We note that the thickness ratio of the ellipse is equal to $(R^2 - 1)/(R^2 + 1)$. It is easy to show that if (4.2) is solved for z as function of ζ, and if this substitution is performed in (4.1), then $w = \phi + i\psi$ yields a symmetric flow past the ellipse with velocity U (not $\frac{1}{2}U$, as previously) at infinity. Furthermore, setting $dw/d\zeta \equiv \dfrac{dw}{dz} \Big/ \dfrac{d\zeta}{dz} = \bar{\mathfrak{q}}$ (i.e., the conjugate of the velocity vector), we can easily show that w may be expressed as a function of $\bar{\mathfrak{q}}$ in the following manner:

$$w = \tfrac{1}{2}U\left[\left(\frac{\bar{\mathfrak{q}} - R^2U}{\bar{\mathfrak{q}} - U}\right)^{\frac{1}{2}} + R^2\left(\frac{\bar{\mathfrak{q}} - U}{\bar{\mathfrak{q}} - R^2U}\right)^{\frac{1}{2}}\right]. \tag{4.3a}$$

Finally, if we set $\bar{\mathfrak{q}} = \exp \mathrm{Z}^{\dagger}$, $\mathrm{Z}^{\dagger} = \lambda^{\dagger} - i\theta$, we obtain the following expression

for the complex potential in the plane of the complex variable $Z^\dagger$ (logarithmic plane):

$$(4.3b)\quad w = g^{[0]}(Z^\dagger) = \tfrac{1}{2} U\left[\left(\frac{\exp Z^\dagger - R^2 U}{\exp Z^\dagger - U}\right)^{\frac{1}{2}} + R^2\left(\frac{\exp Z^\dagger - U}{\exp Z^\dagger - R^2 U}\right)^{\frac{1}{2}}\right].$$

There are two parameters in the flow, namely U, the speed at infinity, and R^2, which determines the thickness ratio of the ellipse. We have chosen $R^2 = 1.2$, corresponding to a thickness ratio of 1/11, while U was chosen equal to .7650; the value of λ at infinity for the compressible flow determined by applying the integral operator (3.6) to the function $g^{[0]}(Z)$ given by (4.3b), will therefore be

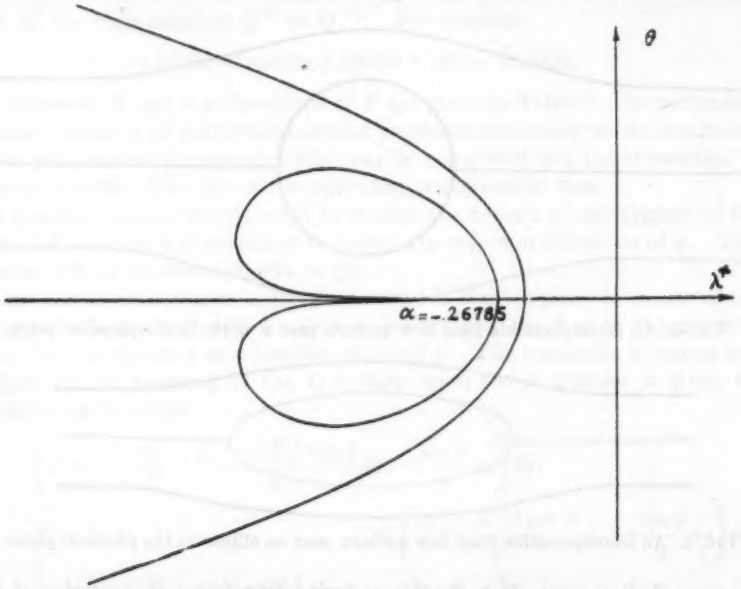

FIG. 4. The image in the logarithmic plane of an incompressible fluid flow pattern past the ellipse

equal to $\alpha = \log(.7650) = -.26785$, corresponding to Mach number 0.6. With these values of the parameters the image of the incompressible flow in the $Z^\dagger$-plane is as indicated in Fig. 4. It should be stressed that the flow is situated on a two-sheeted Riemann surface whose two sheets (corresponding to the upper and lower halves of the physical plane, respectively) are connected along the interval $-\infty < \lambda^\dagger < \alpha$, $\theta = 0$. The point $Z^\dagger = \alpha$ is seen to be a branch point of $g^{[0]}(Z)$.

The function $g^{[0]}(Z)$ was evaluated for a number of values of Z lying within and slightly outside the streamline $\psi = 0$ (See Table IV.0.) Then the function

$$(4.4)\qquad g^{[1]}(Z) = \int_a^Z g^{[0]}(Z')\, dZ', \qquad Z' = \lambda' - i\theta'$$

which can also be expressed in terms of elementary functions,[10] was evaluated for the same set of values of Z. (See Table IV.1) The functions $g^{[3]}(Z)$, $g^{[3]}(Z)$, $g^{[1]}(Z)$, which are defined by the recursion formula:

$$g^{[n+1]}(Z) = \int_{\alpha}^{Z} g^{[n]}(Z')\, dZ' \tag{4.5}$$

were also evaluated for the same set of values of Z. However, since it was not possible to express the functions in closed form a graphical method was employed to evaluate them. The values of $g^{[n+1]}(Z)$ were determined in the following manner: The point Z was joined to the point α by a polygonal path consisting of a vertical segment from Z to the λ-axis and a horizontal segment to the point α. The complex integration (4.5) was thus reduced to two real integrations, for on the vertical portion $dZ' = id\theta'$, while on the horizontal portion $dZ' = d\lambda'$. The

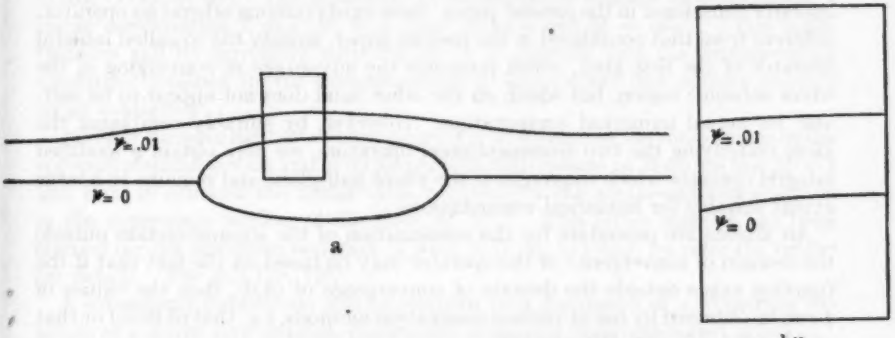

FIG. 5. A compressible fluid flow pattern past an oval-shaped obstacle in the physical plane

integrals were estimated by plotting curves on graph-paper and counting squares. In this manner the functions $g^{[2]}$, $g^{[3]}$, $g^{[4]}$, successively were determined. (See Tables IV.2–IV.4.)

For each of the values of Z under consideration the corresponding values of $H(2\lambda)$, $Q^{(n)}(2\lambda)$, $n = 1, 2, 3, 4$ were determined. These values are listed in Table V.

Then the imaginary part of the series (3.6) which yields the streamfunction of a *compressible* flow[11] (at least in a certain domain of convergence) was approximated by taking the first four terms of the indicated summation. In this manner

[10] We have namely:

$$g^{[1]}(Z) = \tfrac{1}{2} U(1 + R^2) \lg\{[1 + R^2 - 2\exp(Z - \alpha) - 2S]/[R^2 - 1]\} + U R \lg\{[2RS + 2R^2 - (1 + R^2)\exp(Z - \alpha)]/[R^2 - 1]\} - R U(Z - \alpha)$$

where

$$S^2 = \exp[2(Z - \alpha)] - (1 + R^2)\exp(Z - \alpha) + R^2$$

[11] The streamfunction given by (3.6) does not yield a flow past the same obstacle as in the incompressible case, but rather past a more or less distorted obstacle.

[12] The portion of the flow in the plane in Fig. 5a is shown enlarged in Fig. 5b.

the approximate values for the streamfunction listed in Table VI were obtained. Finally, by using formulas (3.7)[13] for the transition to the physical plane, the form of the obstacle and one of the streamlines ($\psi = .01$) were determined approximately. The results obtained are indicated in Tables VII, 1, VII, 2 and Fig. 5.[14]

5. Additional remarks concerning the evaluation of the steamfunction. As we mentioned before, the integral operator in the form (3.6) converges only in a certain portion of the subsonic region (i.e. of the half plane $\lambda < 0$). Since on the other hand flow patterns may be defined partially outside the domains of convergence, the question arises of extending our procedure so as to yield in their entirety (subsonic) flow patterns.[15]

As has been discussed in various papers ([1], [6]), in addition to the integral operator considered in the present paper, there exists (among others) an operator, different from that considered in the present paper, namely the so-called integral operator of the first kind, which possesses the advantage of converging in the whole subsonic region, but which on the other hand does not appear to be suitable for actual numerical computation. However, by suitably combining the ideas underlying the two aforementioned operators, we may obtain a modified integral operator which converges in the whole half-plane and remains to a large extent suitable for numerical computations.

An alternative procedure for the continuation of the streamfunction outside the domain of convergence of the operator may be based on the fact that if the function exists outside the domain of convergence of (3.6), then the values of ψ can be obtained by use of various summation methods, e.g. that of Borel or that of Lindelöf. See [2, §18] and [11].

A second problem important in applications is that of treating mixed flows, i.e., flows which are partially supersonic. If we work in the hodograph plane, or any of the related planes, we have the problem of considering solutions of a linear partial differential equation which is elliptic in one domain (subsonic region) and hyperbolic (supersonic region) in the remaining portion of the domain of definition of the streamfunction under consideration.

[13] The point corresponding to $q = .4571$, $\theta = .348$ (on the zero streamline) was arbitrarily chosen as the origin of the physical plane. The position of the image of the point $q = .4571$, $\theta = .237$ (on the streamline $\psi = .01$) was then determined by using formula (3.7) along the line $q = .4571$. Then the images of the other points on the streamline $\psi = .01$ were determined by performing the integrations indicated in (3.7) along the streamline.

[14] In order to diminish the extent of the tables we give here only a portion of the numerical tabulation, so that the values in Table VII represent data yielding only a portion (indicated in the frame in Fig. 5) of the flow pattern. The determination of the remainder of the flow pattern involves no difficulties.

[15] As it has been indicated in [6, §1 and 8, §6 and §7], in order to determine flow patterns near the transonic line and partially supersonic flows, we need the function $q^{(n,s)}$ in addition to the $Q^{(n)}$. Since at present only the $Q^{(n)}$ are available, in the following we shall discuss methods of getting subsonic flows near the transonic line and mixed flow patterns *without* using the $q^{(n,s)}$.

By introducing integral operators into the theory of partial differential equations of hyperbolic type or by some other means, we can easily obtain streamfunctions of supersonic flow, the other one for a supersonic flow along the line $M = 1$, so that the resulting function defines a mixed flow, which satisfies regularity requirements in the physical plane.

In [2, §17], Bergman has indicated a method of continuing a subsonic flow pattern across the transonic line which consists in considering the compressibility equation in the M, θ-plane and (following Chaplygin [9]) introducing a complete set of solutions $\chi_\nu(M, \theta)$ which are defined for both the subsonic and supersonic region. Naturally, in order to be complete in the neighborhood of the transonic line $\{\chi_\nu\}$ must (in contrast to the set used by Chaplygin) include solutions which are infinite at $M = 0$. For details see [2, §17, in particular formulae (176), (177); 5, §4]. The procedure of continuation consists in determining a linear combination $\Sigma A_\nu \chi_\nu(M, \theta)$ so that along a line $M = M_0 < 1$ this latter function and its derivative with respect to M coincide with the corresponding quantities obtained for the streamfunction of the subsonic flow pattern which is to be continued.

Another possible method of continuation is to determine the limit values of the subsonic streamfunction and of its derivatives with respect to M on the line $M = 1$ and then to consider the initial value problem for the compressibility equation in the supersonic portion of the flow, $M > 1$. This latter method has been studied for the simplified equation of state in [7, 8], to which we refer the reader for details. See also [10, 12].

A subsequent publication will be devoted to a discussion of a numerical example of a mixed flow pattern, determined along the lines mentioned above.

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ON THE DISTRIBUTION OF PLANE STRESS IN A SEMI-INFINITE PLATE WITH PARTIALLY STIFFENED EDGE

By E. L. Buell

1. Introduction. The problem of determining the state of stress in a thin semi-infinite plate subjected to a concentrated load at some point of its unstiffened straight edge was solved by J. Boussinesq in 1892. [1][1] More recently E. Melan [3] has given the solution of a similar problem for a semi-infinite plate to which a stiffening rod is attached along its entire straight edge. In the present paper we shall consider the case where the stiffener is semi-infinite, i.e., it extends to infinity in one direction only from some point of the straight edge at which an axial load is applied to the stiffener.

By means of the theory of conformal mapping the non-uniform boundary problem for the given plate is transformed into a corresponding boundary value problem for the unit circle. For this latter domain a solution is obtained in the form of a Fourier series whose coefficients satisfy an infinite system of linear equations in an infinite number of unknowns. This system is solved approximately and the values of the first few Fourier coefficients thus obtained are used to calculate the stresses at the edge of the plate. The results are then compared with those of Boussinesq and Melan to show the effects of the stiffener and to derive an "effective length" for it.

2. Formulation of the problem. We treat the problem as one in "plane stress" with the cross sectional area A of the stiffener assumed concentrated along a line. Fig. 1 shows the semi-infinite sheet with the stiffener attached along its straight edge from the origin to infinity in the positive x direction. Also shown are the stresses σ_x, σ_y, and τ acting on an element of the sheet. The stresses are functions of the point coordinates (x and y) and are produced by the constant load P applied to the end of the stiffener in a direction parallel to the positive x axis. They must satisfy the following boundary conditions:

Since it is assumed that there is no force acting normal to the edge of the plate,

$$\sigma_y(x, 0) = 0 \text{ for all } x. \tag{1}$$

Since there is no force acting parallel to the edge of the plate along its unstiffened part,

$$\tau(x, 0) = 0 \text{ for } x < 0. \tag{2}$$

The boundary condition along the stiffened part of the edge is obtained by considering the equilibrium of the forces acting on any portion of the stiffener. Assuming that Hooke's law applies and that there is no slipping between the stiffener and the plate, the following equation can be established:

$$\sigma_x(x, 0) = -(E_s/AE_m)\left(P + t\int_0^x \tau(\xi, 0)\, d\xi\right). \quad (x > 0) \tag{3}$$

[1] The numbers in brackets refer to the bibliography.

Here E, and E_{st} are the Young's moduli for the plate and stiffener, respectively, t is the (constant, small) thickness of the plate, and ξ is a running coordinate in the x direction.

Equations (1), (2), and (3), together with (7), form the complete boundary conditions along the straight edge of the plate. The requirements at infinity are that all the stresses must approach zero.

In terms of the Airy stress function $\Phi(x, y)$ the stresses are given by[2]

$$\sigma_x(x, y) = \Phi_{yy}(x, y), \qquad \sigma_y(x, y) = \Phi_{xx}(x, y), \qquad \tau(x, y) = -\Phi_{xy}(x, y). \quad (4)$$

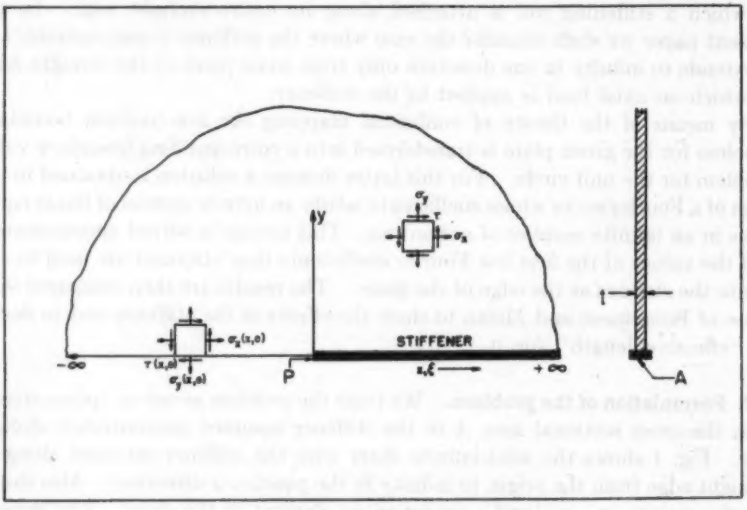

FIG. 1

From the stress-strain relations it follows that Φ must be a solution of the bi-Laplace equation $\nabla^2\nabla^2\Phi = 0$ at all points of the plate. The most convenient form of its general solution for the present purpose is [4]

$$\Phi = y\varphi + \varphi', \quad (5)$$

where φ and φ' are potential functions, since the first term drops out along the edge of the plate ($y = 0$). The problem is now reduced to finding the functions φ and φ' which satisfy the boundary conditions stated above.

From (4) and (5), $\sigma_y = y\varphi_{xx} + \varphi'_{xx}$, and hence, by (1), $\varphi'_{xx}(x, 0) = 0$. Integration gives $\varphi'(x, 0) = c_1x + c_2$, but since the addition of a linear function to Φ will not affect the stresses, it is permissible to set $c_1 = c_2 = 0$. Hence the boundary condition (1) is satisfied by taking $\varphi' \equiv 0$.

Also from (4) and (5) it follows that $\tau = -(y\varphi_{xy} + \varphi_x + \varphi'_{xy})$, so that

$$\tau(x, 0) = -\varphi_x(x, 0), \quad (6)$$

[2] See for example [2]. Subscripts on Φ indicate partial derivatives.

which is zero for negative x by (2). A further condition on τ results from the equilibrium requirement between the external load and the total shear applied to the stiffener along the edge of the plate. This condition is

$$P = -\int_{-\infty}^{\infty} \tau(x, 0)t\, dx = -t\int_{0_-}^{\infty} \tau(x, 0)\, dx. \tag{7}$$

The lower limit is written 0_- to allow for the possibility that τ may be infinite at $x = 0$. However, the integral must exist, which shows that the order of the singularity in $\tau(x, 0)$ at the origin must be less than one and that, as x becomes infinite, $\tau(x, 0)$ approaches zero more rapidly than x^{-1}.

Integration of (6) gives

$$\varphi(x, 0) = C = 0 \text{ for } x < 0, \tag{8}$$

where the constant of integration may be set equal to zero since the stresses will not be affected by the addition of a constant to φ.

Using (6), (7) becomes

$$P = t\int_{0_-}^{\infty} \varphi_x(x, 0)\, dx, \tag{9}$$

or

$$\lim_{h\to\infty} \varphi(h, 0) - \varphi(0_-, 0) = \bar{P}, \tag{10}$$

where $\bar{P} = P/t$. In view of (8), (10) reduces to

$$\lim_{h\to\infty} \varphi(h, 0) = \bar{P}. \tag{11}$$

Finally, from (4) and (5),

$$\sigma_x = y\varphi_{yy} + 2\varphi_y + \varphi'_{yy} \quad (12), \qquad \text{and } \sigma_x(x, 0) = 2\varphi_y(x, 0). \tag{13}$$

Substituting (6) and (13) in (3) and integrating, we obtain

$$\varphi(x, 0) - 2L\varphi_y(x, 0) = \bar{P}, \qquad (x > 0) \tag{14}$$

where $L = AE_{st}/tE_s$. The problem has thus been reduced to the determination of the function $\varphi(x, y)$ which satisfies the Laplace equation ($\nabla^2\varphi = 0$) for the upper half plane ($y > 0$) and the boundary conditions (8), (11), and (14) along the line $y = 0$. It should be noted that $\varphi(0_+, 0) = 0$, which follows from (3) and (14) when $x \to 0$. Hence $\varphi(x, 0)$ will be continuous at the origin if its value there is defined to be zero.

3. Solution of the boundary value problem. Let the plane of the plate be taken as the complex z plane, where

$$z = x + iy = re^{i\theta}. \tag{15}$$

This entire plane is mapped conformally into the interior of the unit circle with center at the origin of a complex ζ plane by means of the analytic function

$$\zeta = [i - (z/L)^{\frac{1}{2}}]/[i + (z/L)^{\frac{1}{2}}]. \tag{16}$$

By this transformation the "upper half" ($y > 0$) of the z plane maps into the "upper half circle"[3] of the ζ plane, and the "lower half" ($y < 0$) into the "lower half circle." It is convenient to imagine the z plane as slit along its positive x axis, the "upper edge" ($\theta = 0$) of which maps into the "upper semicircle" ($\rho = 1, 0 < \psi < \pi$) of the ζ plane ($\zeta = \rho e^{i\psi}$) and the "lower edge" ($\theta = 2\pi$) into the "lower semicircle" ($\rho = 1, \pi < \psi < 2\pi$). The negative x axis maps into the segment $-1 < Re\{\zeta\} < 1$ of the real axis in the ζ plane.

Next the boundary conditions must be mapped onto the ζ plane. To that end consider the analytic function

$$f(z) = \varphi(x, y) + i\chi(x, y), \tag{17}$$

in terms of which

$$\varphi_x = Re\{df/dz\} \text{ and } \varphi_y = -Im\{df/dz\}. \tag{18}$$

Writing $df/dz = (df/d\zeta)(d\zeta/dz)$ and treating each factor separately, a straightforward calculation yields

$$\overline{\varphi_x} = \frac{\cos^3 \tfrac{1}{2}\psi}{L \sin \tfrac{1}{2}\psi} \overline{\frac{\partial \varphi_1}{\partial \psi}} \quad \text{and} \quad \overline{\varphi_y} = -\frac{\cos^3 \tfrac{1}{2}\psi}{L \sin \tfrac{1}{2}\psi} \overline{\frac{\partial \varphi_1}{\partial \rho}}, \tag{19}$$

where $\varphi_1(\rho, \psi)$ is the potential function into which $\varphi(x, y)$ is transformed by (16) and where the bar indicates values on the boundary ($x > 0, y = 0$ in the z plane; $\rho = 1$ in the ζ plane). That is,

$$\varphi(x, y) = \varphi_1(\rho, \psi) \text{ and for } x > 0, \qquad \bar{\varphi} = \varphi(x, 0) = \varphi_1(1, \psi) = \bar{\varphi}_1. \tag{20}$$

These results show that the boundary conditions (8), (11), (14) become, when transformed to the ζ plane,

$$\varphi_1(\rho, \psi) = 0 \text{ for } -1 < Re\{\zeta\} < 1, \tag{21}$$

$$\lim_{\psi \to \pi -} \varphi_1(1, \psi) = \lim_{\psi \to \pi -} \bar{\varphi}_1 = \bar{P}, \tag{22}$$

$$\bar{\varphi}_1 + 2 \frac{\cos^3 \tfrac{1}{2}\psi}{\sin \tfrac{1}{2}\psi} \overline{\frac{\partial \varphi_1}{\partial \rho}} = \bar{P}. \qquad (0 < \psi < \pi) \tag{23}$$

It should be noted that equations (21), (22), and (23) apply to the upper semicircle of the ζ plane. Since the behavior of $\varphi_1(\rho, \psi)$ outside this domain is arbitrary, we shall, for convenience, define it for the lower half circle by means of the relation

$$\varphi_1(\rho, \psi) = -\varphi_1(\rho, -\psi). \tag{24}$$

In thus making φ_1 an odd function of ψ, equation (21) is satisfied and φ_1 may be expanded in a Fourier series of sine terms only. The boundary conditions on the lower semicircle are obtained from (22) and (23) by replacing ψ by $-\psi$ and

[3] By a "half circle" is meant the area bounded by a semi-circle and the diameter through its ends.

φ_1 by $-\varphi_1$ and will be satisfied by the φ_1 which satisfies (22), (23), and (24). Applied to (22) this process yields

$$\lim_{\psi \to -\pi_+} \varphi_1(1, \psi) = \lim_{\psi \to -\pi_+} \bar{\varphi}_1 = -\bar{P}, \tag{25}$$

which with (22) shows that $\varphi_1(\rho, \psi)$ has a discontinuity of magnitude $2\bar{P}$ at $(1, \pi)$.

Instead of expanding φ_1 directly in a Fourier sine series, better convergence may be expected if this known discontinuity is first removed. To this end let

$$\varphi_1(\rho, \psi) = \varphi_2(\rho, \psi) + \varphi_3(\rho, \psi) \tag{26}$$

where

$$\varphi_3(\rho, \psi) = \sum_{n=1}^{\infty} b_n \rho^n \sin n\psi \tag{27}$$

and the simplest choice for φ_2 on the boundary is

$$\varphi_2(1, \psi) = \bar{\varphi}_2 = \bar{P}\psi/\pi, \qquad (-\pi < \psi < \pi) \tag{28}$$

which is odd and has the same discontinuity as φ_1. If φ_2 is expressed as a Fourier sine series and the latter summed, it turns out that

$$\varphi_2(\rho, \psi) = (2\bar{P}/\pi)\, Im\{ln(1 + \rho e^{i\psi})\}, \tag{29}$$

from which it follows that

$$\overline{\partial\varphi_2}/\partial\rho = (\bar{P}/\pi) \tan \tfrac{1}{2}\psi. \tag{30}$$

Substituting (27), (28), and (30) in (23), the boundary equation becomes

$$\sum_{n=1}^{\infty} (1 + 2n \cos^3 \tfrac{1}{2}\psi \csc \tfrac{1}{2}\psi) b_n \sin n\psi = (\bar{P}/\pi)(\pi - \psi - 2 \cos^2 \tfrac{1}{2}\psi). \tag{31}$$

Multiplying both sides by $\sin \frac{1}{2}\psi \sin m\psi$, where m is a positive integer, and integrating over $(0, \pi)$, we obtain an infinite number of equations in the infinite number of unknowns b_n of the form

$$\sum_{n=1}^{\infty} \alpha_{mn} b_n = \beta_m \qquad (m = 1, 2, 3, \cdots) \tag{32}$$

where

$$\alpha_{mn} = n\left\{\frac{24n(-1)^{n+m}[4(m^2 + n^2) - 5] - [4(m+n)^2 - 9][4(m-n)^2 - 9]}{[4(m+n)^2 - 1][4(m-n)^2 - 1][4(m+n)^2 - 9][4(m-n)^2 - 9]}\right\} \tag{33}$$

and

$$\beta_m = \begin{cases} -\dfrac{8\bar{P}}{\pi(4m^2 - 1)^2(4m^2 - 9)} & \text{if } m \text{ is even} \\ \dfrac{2\bar{P}(4m^2 - 5)}{\pi(4m^2 - 1)^2(4m^2 - 9)} & \text{if } m \text{ is odd.} \end{cases} \tag{34}$$

Analysis of these results shows that the b_n are approximately $0(n^{-4})$ as $n \to \infty$ and that the signs alternate, so that the convergence is very rapid. In passing it may be noted that if φ_1 is expanded in a Fourier sine series without removing the discontinuity, the coefficients are found to be only about $0(n^{-2})$, thus verifying the advantage of taking out the discontinuity first.

Since $|\alpha_{mn}|$ decreases rapidly along any row or column with increasing distance from the principal diagonal, an approximate solution of the system (32) may be effected by replacing it by the finite system

$$\sum_{n=1}^{p} \alpha_{mn} b_n = \beta_m \qquad (m = 1, 2, 3, \cdots, p) \quad (35)$$

for $p = 1, 2, 3, \cdots$. The results of this procedure[4] up to $p = 6$ are shown in table 1. Although the convergence of this process of successive approximations

TABLE 1

Values of $(\pi/\overline{P})b_n$

	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$
$n = 1$	.311111	.365874	.375729	.376277	.376186	.376141
$n = 2$		−.094782	−.121326	−.124580	−.125028	−.125031
$n = 3$			.044485	.052465	.054284	.054454
$n = 4$				−.012464	−.016478	−.017024
$n = 5$					.005922	.007033
$n = 6$						−.001567

has not been established analytically, it is evident physically and is suggested by a comparison of the values of $(\pi/\overline{P})b_n$ tabulated in successive columns of table 1. A further approximate check can be obtained by satisfying (31) exactly at k equally spaced values of ψ inside $(0, \pi)$, terminating the series at $n = k$, and solving the resulting simultaneous linear equations in $b_1, b_2, \cdots, b_k$. For $k = 5$ this process yields values differing numerically from those found above by less than 0.0016, thereby indicating sufficient accuracy for numerical calculations.

Having thus determined the first few b_n, we may regard φ_2 and φ_1 as sufficiently established. There remains the problem of calculating the desired stresses at the edge of the plate in terms of φ_1 and thence as functions of x by reversing the mapping.

[4] The solution of the systems of simultaneous linear equations was greatly facilitated by Crout's method [5], which is particularly well suited for use with an ordinary computing machine. It was especially convenient here, since the so-called auxiliary matrix computed for the case $p = i$ carried over to form all but one row and one column of the auxiliary matrix needed for the next case, vis., $p = i + 1$.

4. Calculation of the edge stresses. The desired stresses at the edge of the plate are $\sigma_x(x, 0)$ for all x and $\tau(x, 0)$ for $x > 0$, all others being zero as stated in (1) and (2).

Consider first $\sigma_x(x, 0)$ for $x > 0$. From (13), (14), and (20) we have

$$L\sigma_x(x, 0) = \phi_1 - \bar{P}, \tag{36}$$

which becomes, using (26), (27), and (28),

$$(L/\bar{P})\sigma_x(x, 0) = [1/\pi]\left[\psi - \pi + \sum_{n=1}^{\infty} (\pi b_n/\bar{P}) \sin n\psi\right]. \tag{37}$$

From the mapping function (16) it can be shown that, for $y = 0$ and $x > 0$,

$$\psi = \arctan\{2(x/L)^{\frac{1}{2}}/(1 - x/L)\}, \tag{38}$$

$$\sin n\psi = (-1 - x/L)^{-n} Im\{[i - (x/L)^{\frac{1}{2}}]^{2n}\}, \tag{39}$$

$$\tan \tfrac{1}{2}\psi = (x/L)^{\frac{1}{2}}, \tag{40}$$

$$\cos n\psi = (-1 - x/L)^{-n} Re\{[i - (x/L)^{\frac{1}{2}}]^{2n}\}. \tag{41}$$

Equations (40) and (41) will be needed shortly, while (38) and (39) may be substituted in (37) to give, for $x > 0$,

$$(L/\bar{P})\sigma_x(x, 0) = [1/\pi]\left[\arctan\{2(x/L)^{\frac{1}{2}}/(1 - x/L)\} - \pi + \sum_{n=1}^{\infty} (-1 - x/L)^{-n}(\pi b_n/\bar{P}) Im\{[i - (x/L)^{\frac{1}{2}}]^{2n}\}\right]. \tag{42}$$

An alternative expression for $\sigma_x(x, 0)$ can be obtained from (13), using (19), (26), (27), (30), (39), and (40). The result is

$$(L/\bar{P})\sigma_x(x, 0) = -[2/\pi]\left[(1 + x/L)^{-1} - (x/L)^{-\frac{1}{2}} \sum_{n=1}^{\infty} (-1 - x/L)^{-n-1} \cdot (n\pi b_n/\bar{P}) Im\{[i - (x/L)^{\frac{1}{2}}]^{2n}\}\right]. \quad (x > 0) \tag{43}$$

The convergence of the series in (42) would be expected to be better than that in (43) owing to the factor n in the latter.[5]

An expression for $\tau(x, 0)$ when $x > 0$ may be found in a similar fashion from (6), (19), (26), (27), (28), and (41). The result is

$$(L/\bar{P})\tau(x, 0) = -[\pi(x/L)^{\frac{1}{2}}(1 + x/L)]^{-1}\left[1 + \sum_{n=1}^{\infty} (-1 - x/L)^{-n}(n\pi b_n/\bar{P}) \cdot Re\{[i - (x/L)^{\frac{1}{2}}]^{2n}\}\right]. \quad (x > 0) \tag{44}$$

[5] It can be shown that all series appearing in (42), (43), (44), (45), and (46) converge absolutely and uniformly if the $b_n \to 0$ faster than $0(n^{-2})$. Since the discussion following (34) shows this condition to be satisfied, the use of the series and the operations performed upon them are justified.

Since differentiation of (3) yields

$$\tau(x, 0) = -L\overline{\partial\sigma_x}/\partial x, \qquad (x > 0) \quad (45)$$

$\tau(x, 0)$ is proportional to the slope of the $\sigma_x(x, 0)$ curve and two alternative formulas for $\tau(x, 0)$ may be obtained by differentiating (42) and (43), but again the convergence of the latter would presumably be slower than that of (44).

To obtain $\sigma_x(x, 0)$ for $x < 0$, it is necessary to map (13) onto the ζ plane, i.e., to transform $\varphi_y(x, 0)$ for $x < 0$ in a manner analogous to that used for $x > 0$ in deriving (19), recalling that $y = 0, x < 0$ implies $0 < \rho < 1$ and $\psi = 0$ or π

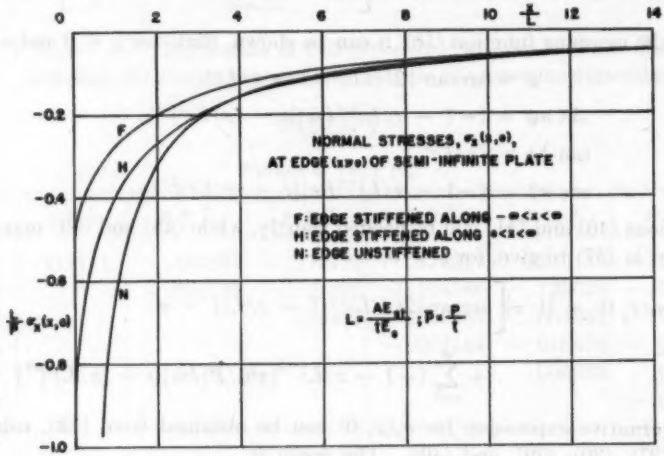

FIG. 2

The resulting expression is

$$(L/\overline{P})\sigma_x(x, 0) = (2/\pi)(-x/L)^{-\frac{1}{2}}[1 + (-x/L)^{\frac{1}{2}}]^{-1}\left\{1 + \sum_{n=1}^{\infty}[n\pi b_n/\overline{P}]\cdot[1 - (-x/L)^{\frac{1}{2}}]^{n-1}[1 + (-x/L)^{\frac{1}{2}}]^{-n}\right\}. \qquad (x < 0) \quad (46)$$

By using only the first five terms of the series in (42), (44), and (46), together with the values of the first five $(\pi/\overline{P})b_n$ in the last column of table 1 rounded to the nearest four decimal places, the values of $\sigma_x(x, 0)$ and $\tau(x, 0)$ corresponding to assigned values of x/L can be computed. These results are shown graphically in figs. 2, 3, and 4 by the curves labelled "H" (for "half stiffener").

5. Analysis and comparison of results. The behavior of the edge stresses near $x = 0$ and $x = \pm\infty$ is of special interest and may be determined from the formulas given above. From (42) we have

$$\sigma_x(x, 0) \to -\overline{P}/L = -PE_o/AE_{st} \text{ as } x \to 0_+, \qquad (47)$$

which agrees with (3). From (43) we also have

$$\sigma_x(x, 0) \to -2\bar{P}/\pi x \text{ as } x \to +\infty. \tag{48}$$

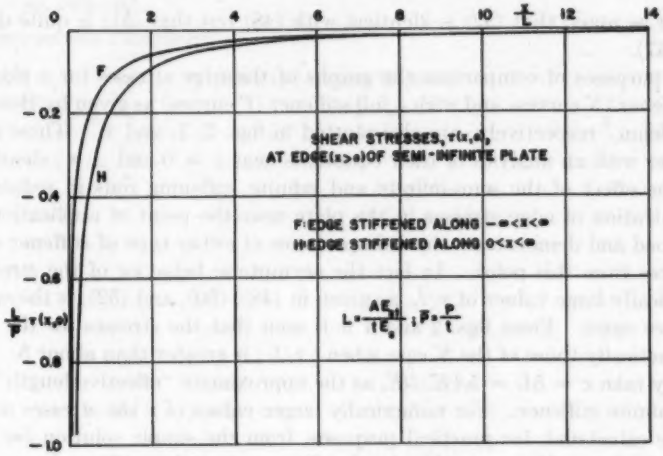

FIG. 3

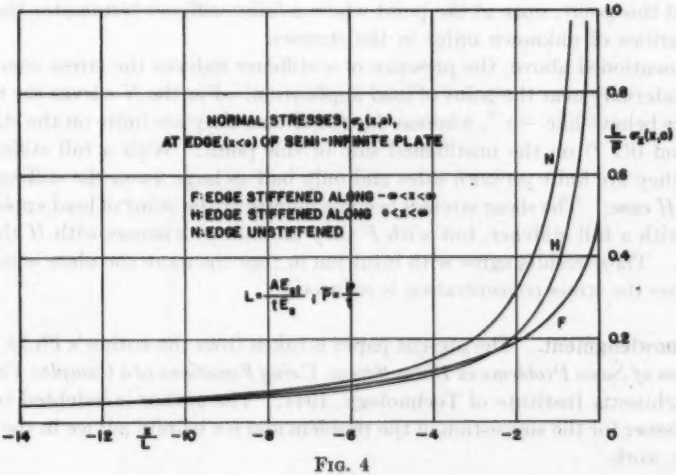

FIG. 4

Similarly the behavior of $\tau(x, 0)$ is found from (44) and (45) to be

$$\tau(x, 0) = 0(x^{-\frac{1}{2}}) \text{ as } x \to 0_+, \tag{49}$$

$$\tau(x, 0) \to -2\bar{P}L/\pi x^2 = -2APE_s t/\pi E_s t^2 x^2, \text{ as } x \to +\infty, \tag{50}$$

which makes precise the bounds stated in the discussion following (7).

Similar analysis of (46) shows that

$$\sigma_x(x, 0) = 0[(-x)^{-\frac{1}{2}}] \text{ as } x \to 0_-, \tag{51}$$

and

$$\sigma_x(x, 0) \to -2\bar{P}/\pi x \text{ as } x \to -\infty. \tag{52}$$

It is to be noted that (52) is identical with (48) but that (51) is quite different from (47).

For purposes of comparison the graphs of the edge stresses for a plate with *no* stiffener (N curves) and with a *full* stiffener (F curves) as given by Boussinesq and Melan,[6] respectively, are also plotted in figs. 2, 3, and 4.[7] These graphs, together with an analysis of their equations near $x = 0$ and $\pm\infty$, clearly indicate the effect of the semi-infinite and infinite stiffening rods in reducing the concentration of edge stresses in the plate near the point of application of an axial load and demonstrate the unimportance of either type of stiffener at large distances from this point. In fact the asymptotic behavior of the stresses for numerically large values of x/L, as given in (48), (50), and (52), is the same for all three cases. From figs. 2 and 4 it is seen that the stresses for the H case are practically those of the N case when $|x/L|$ is greater than about 5. Hence we may take $x = 5L = 5AE_s/tE_s$ as the approximate "effective length" of the semi-infinite stiffener. For numerically larger values of x the stresses might as well be calculated, for practical purposes, from the simple solution for an unstiffened plate. However, this does not mean that the stiffener has no effect beyond this point, since at the point where a finite stiffener terminates there are singularities of unknown order in the stresses.

As mentioned above, the presence of a stiffener reduces the stress concentration materially near the point of load application. For the N curves the normal stresses behave like $-x^{-1}$, whereas for the H case they are finite on the stiffened side and $0(x^{-\frac{1}{2}})$ on the unstiffened side of this point. With a full stiffener (F case) they are finite on both sides and only half as large as on the stiffened side of the H case. The shear stresses become infinite at the point of load application even with a full stiffener, but with F they are $0(\ln x)$ whereas with H they are $0(x^{-\frac{1}{2}})$. These results agree with intuition in that the more the plate is stiffened the more the stress concentration is reduced.

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[6] See [1], [2], and [3].

[7] Fig. 3 has no N curve, since the shear stress for the unstiffened plate is zero along the edge except at the origin where it is infinite.

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COMPLETE ELLIPTIC INTEGRALS OF COMPLEX LEGENDRIAN MODULUS[1]

By Enzo Cambi

1. Complete elliptic integrals (as well as some elliptic functions) are sometimes of interest for complex values of the Legendrian modulus k. In the following, it is preferred to regard them as functions *of the square* of the modulus, $z = k^2$, by writing

$$K(z) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - z\sin^2\varphi}}; \qquad E(z) = \int_0^{\pi/2} \sqrt{1 - z\sin^2\varphi}\, d\varphi.$$

We denote as usual by K' and E' the integrals of the *complementary modulus* k' $(k'^2 + k^2 = 1)$:

$$K'(z) = K(1 - z); \qquad E'(z) = E(1 - z).$$

In terms of the (complex) variable z, the integrals K and K' (which are the most important for applications) satisfy the hypergeometric equation[2]

$$z(1 - z)\, d^2y/dz^2 + (1 - 2z)\, dy/dz - \tfrac{1}{4}y = 0. \tag{1}$$

Similarly

$$z(1 - z)\frac{d^2E}{dz^2} + (1 - z)\frac{dE}{dz} + \frac{1}{4}E = 0;$$

$$z(1 - z)\frac{d^2E'}{dz^2} - z\frac{dE'}{dz} + \frac{1}{4}E' = 0.$$

Rather than in problems of actual quadrature, integrals of complex modulus are likely to occur in connection with the solution of linear differential equations, of types allied to the above. We are led to equations of this kind, for instance, when transforming into algebraic form differential equations, having rational functions of *sin* x and *cos* x as coefficients, under the assumption of a rational function of e^{ix} as a new independent variable: a substitution that may be of interest when, owing to the presence of real singularities, the solution of the *periodic* equation in Floquet's form is not practicable.

2. The singularities of a differential equation having periodic coefficients are obviously situated (owing to the periodicity itself) on straight lines parallel to the real axis. Hence, the periodic function $P(x)$ appearing in Floquet's expression of the integrals, $y = e^{\mu x} P(x)$, when expressed as a Fourier series, will converge in the strip bounded by the two parallels to the real axis, passing

[1] For general reference, see, for instance, E. T. Whittaker—G. N. Watson—*Modern Analysis*—Cambridge 1935, Chapters XX (Elliptic Functions), and XIX (for periodic differential equations).

[2] Whittaker-Watson, page 499. For a complete discussion of the equation, see J. Tannery—J. Molk—*Fonctions Elliptiques*—Paris 1898—III vol.—page 188 *et seq.*

through the singularities whose imaginary parts, respectively negative and positive, are the smallest in absolute value.

If the equation (after reduction of the coefficient of the highest differential quotient to unity) has *integral* coefficients, the solutions are also integral functions, and the Fourier series in Floquet's solutions converge in the whole plane. Considering, for instance, the second order equations, this is the case for Hill's (including Mathieu's) equation. In such a case, the transformation of the coefficients into algebraic form is not convenient, since the Floquet's solutions are then valid for any value of x.

When the equation has real singularities, on the contrary, Hill's method of solution is not applicable (unless the singular coefficients are *bounded* in their singularities; but the convergence of the method is very slow in any case); the same is true, whenever the singularities are *finite*, outside the strip of convergence. The simplest second order equation having finite singularities can be assumed as[3]

$$(a + b\cos x)\, d^2y/dx^2 + y = 0. \qquad (a, b \text{ real}) \tag{2}$$

The equation has, or has not, real singularities, according as $|b| \gtrless |a|$. The limiting case is $|b| = |a|$, where the equation has a *single* real singularity in every period.

The case $a = b = 8$ has a peculiar interest. When this is the case, the equation

$$8(1 + \cos x)\, d^2y/dx^2 + y = 0$$

which can also be written

$$4\cos^2 u\, d^2y/du^2 + y = 0 \qquad [u = \tfrac{1}{2}x],$$

is readily transformed, by the substitution $-e^{ix} = t$, into the Legendre equation

$$t(1-t)^2\, d^2y/dt^2 + (1-t)^2\, dy/dt + \tfrac{1}{4}\, y = 0. \tag{3}$$

The equation is, substantially, the same as (1), to which it can be reduced by either assuming $y(1-t)^{-\frac{1}{2}}$ as a new unknown function, or by transforming the independent variable t through the involutory substitution $t = z/(z-1)$; $(z = t/(t-1))$. Two independent integrals of (3) may hence be written as $\sqrt{1-t}\, K(t)$ and $\sqrt{1-t}\, K'(t)$, or as $K(t/(t-1))$, $K'(t/(t-1))$.

When $a = b \neq 8$, the same substitution still leads to a hypergeometric equation that can be solved in terms of complete integrals involving elliptic functions of complex modulus. In view of the greater simplicity of the conclusions, however, the equation $a = b = 8$ is of more immediate interest.

3. The substitution $t = \pm e^{ix}$ (or, generally, t equal to a rational function of e^{ix}) establishes a $(1, \infty)$ correspondence between t and x. Consequently, the x-plane cannot be put in correspondence with a *single* t-plane, but has to be referred to a Riemann surface constituted by an infinite number of t-sheets,

[3] An account of the solution of equation (2) is given in two papers by the author, in *Atti dell'Acc. Naz. dei Lincei* Nos. 10–11, 1946, p. 1035 and 1181.

connected with one another along the edges of cuts, joining the *introduced* singularities $t = 0$ and $t = \infty$ and the *actual* singularities of the considered function of x, if any. Each sheet of such Riemann surface, covering the whole t-plane, can be put in (1, 1) correspondence with a strip of the x-plane having a width 2π in the direction of the real axis.

The substitution transforms the real x-axis into a line of the t-plane encircling both the introduced singularities. In the case of the substitution $t = -e^{ix}$, which leads from equation (2) to the hypergeometric equation (3), the real x-axis corresponds to the circle of radius unity, having its center in the origin (*unit circle* 0). The standard homographic substitutions of the hypergeometric equation ($t = 1 - z$, or $1/z$, or $z/(z-1)$, or $1/(1-z)$) in turn, either transform this line into itself, or into one of the following: circle of radius unity, center in $z = 1$ (*unit circle* 1); straight line 'real part of $z = \frac{1}{2}$' ($R(z) = \frac{1}{2}$).

4. In a complex z-plane, the above defined lines are of particular interest. They bound the regions of convergence of the hypergeometric series giving the solutions of the hypergeometric equation around the singular points 0, 1, and ∞. The three lines intersect each other in the common points $z = \frac{1}{2} \pm \frac{1}{2}i\sqrt{3}$.

In connection with the solution of linear differential equations of the types considered above, we can then be led to the evaluation of the elliptic integrals, or of more complicated functions of the modulus, along either of these lines. The present paper deals summarily with such computations; numerical values of the most important functions are given in skeleton tables.

5. When k (i.e., z) is real, a very elegant and convenient method of computation of the complete elliptic integrals is that of Gauss' arithmetico-geometrical mean (a.g.m.)[4]. The method owes its high practical value to the following features: i) its convergence is exceptionally rapid; ii) the method is, to some extent, self-checking, in the understanding that trivial errors are almost impossible in the computation of the a.g.m.

Starting with two numbers a_0 and b_0, and computing

$$a_1 = \tfrac{1}{2}(a_0 + b_0); \qquad b_1 = \sqrt{a_0 b_0},$$

and so on, we construct two successions of numbers a_n and b_n, which converge, with extraordinary rapidity, to a common limit $a_\infty = b_\infty$, which is the arithmetico-geometrical mean between a_0 and b_0.

If the initial values are $a_0 = 1$, and $b_0 = k' = \sqrt{1 - k^2}$ the a.g.m. is simply connected with the elliptic integral K through

$$K(k^2) = \pi/(2a_\infty).$$

[4] A very complete discussion of the methods of computation of the elliptic functions based on the a.g.m. scale is given in: L. V. King—*On the direct numerical calculation of elliptic functions and integrals*—Cambridge 1924. In the following, this work is simply referred to as 'King'. W. Bartky (Reviews of Modern Physics—10, p. 264, 1938) discusses a remarkable extension of the method to the computation of 'complete' integrals of a very general class, including Legendre's complete integrals of the first, second, and third kind as particular cases.

Other elliptic integrals, complete or incomplete, as well as other functions of the modulus k, are also computed in terms of the (a_n, b_n) scale, which is conveniently completed by computing the terms $c_n = \frac{1}{2}(a_{n-1} - b_{n-1})[c_n^2 = a_n^2 - b_n^2]$.

6. When $z = k^2$ is generally complex, the elliptic integrals of the first kind K and K' can be defined, as solutions of the hypergeometric equation (1), by the series

$$K(z) = \frac{\pi}{2}F\{\tfrac{1}{2}, \tfrac{1}{2}; 1; z\} = \frac{\pi}{2}\left\{1 + \left(\frac{1}{2}\right)^2 z + \left(\frac{1.3}{2.4}\right)^2 z^2 + \cdots\right\}$$

$$K'(z) = -\frac{1}{\pi}K(z)\log\frac{z}{16} - 2\left\{\left(\frac{1}{2}\right)^2(1 - \tfrac{1}{2})z + \left(\frac{1.3}{2.4}\right)^2(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4})z^2 + \cdots\right\},$$

which converge in the inside of the unit circle 0. By replacing z by $1 - z$ we obtain the series $K(1 - z) = K'(z)$ and $K'(1 - z) = K(z)$ converging *in* the unit circle 1; similarly, by means of the other substitutions which do not change the hypergeometric equation, the functions can be *continuated* in any region of the z-plane bounded by either of the three lines described above.

Owing to the simple relations existing between points z belonging to these lines, the behaviour of the elliptic functions for k^2 lying *on* either of them is known, as soon as it has been determined, for instance, on the unit circle 0.

When $|z| = 1$, we write $z = -e^{ix} = e^{-iy}$ (x real, $y = \pi - x$). The evaluation of $K(z)$ could then be performed by computing the a.g.m. scale $a_0 = 1$, $b_0 = (1 + e^{ix})^{\frac{1}{2}}$, whose terms are complex. The computation of a complex a.g.m. scale is actually somewhat longer than in the case of real terms, since each term has now to be computed both in binomial and exponential form; each of the forms being required for the evaluation of the next arithmetical, or geometrical mean, respectively. The attainable accuracy is obviously limited by that of the trigonometrical table employed in the computation of the square roots. The rapidity of convergence of the scale, however, is so satisfactory, that the a.g.m. can even be used conveniently in the case of complex terms[5]: in the computation of the mean $(1, \sqrt{1 + e^{ix}})$, for instance, so long as $|x| \leq \pi$, a_3 and b_3 already agree with one another, in modulus and phase, to within seven places.

The evaluation of complex terms can be avoided, however, in the computation of K' and K. When computing K', i.e., the a.g.m. $a_0 = 1$, $b_0 = k = e^{-iy/2}$, we note that all of the terms of the scale which follow the first ones, have the same phase. Actually, when $b_0 = e^{-iy/2}$, both $a_1 = \frac{1}{2}(a_0 + b_0)$ and $b_1 = \sqrt{b_0}$ have the phase $-y/4$, which remains unchanged when evaluating the successive terms; these can then be computed by simply operating on their moduli.

Since $K'(k^2) = \pi/(2a_\infty)$, the phase of K' is $+y/4$. Further, since $|b_1| = 1$,

[5] Of course, in order that the procedure may converge, the square roots defining every b-term are to be so determined that their representative point may belong to the same half-plane as the corresponding term a.

and $|a_1| = \cos(y/4)$, the a.g.m. $(1, b_0)$ is equal in modulus to that $(1, \cos \frac{1}{4}y)$:

$$K'(k^2) = K'(e^{-iy}) = e^{iy/4}K(\sin^2\tfrac{1}{4}y)$$

TABLE I

K' for $k^2 = -e^{ix} = e^{-iy}$; or K for $k^2 = 1 + e^{ix} = 1 - e^{-iy}$

x	y	MODULUS	δ^3	δ^4	δ^5	PHASE
0	2ᴸ	1.85407 46773	+286 25285	+4 00042	+19620	+.5ᴸ
.1ᴸ	1.9	1.82219 31420	262 40651	3 31172	14623	.475
.2	1.8	1.79293 56718	241 87189	2 76925	11094	.45
.3	1.7	1.76609 69205	224 10652	2 33772	8489	.425
.4	1.6	1.74149 92344	208 67887	1 99108	6616	.4
.5	1.5	1.71898 83370	195 24230	1 71060	5133	.375
.6	1.4	1.69842 98626	183 51633	1 48145	4190	.35
.7	1.3	1.67970 65515	173 27181	1 29420	3239	.325
.8	1.2	1.66271 59585	164 32149	1 13934	2706	.3
.9	1.1	1.64736 85804	156 51051	1 01154	2173	.275
1.0	1.0	1.63358 63074	149 71107	90547	1837	.25
1.1	.9	1.62130 11451	143 81710	81777	1496	.225
1.2	.8	1.61045 41538	138 74090	74503	1292	.2
1.3	.7	1.60099 45715	134 40973	68521	1100	.175
1.4	.6	1.59287 90865	130 76377	63639	970	.15
1.5	.5	1.58607 12392	127 75420	59727	832	.125
1.6	.4	1.58054 09339	125 34190	56647	767	.1
1.7	.3	1.57626 40476	123 49607	54334	714	.075
1.8	.2	1.57322 21220	122 19358	52735	642	.05
1.9	.1	1.57140 21322	121 41844	51778	651	.025
2	0	1.57079 63268 = $\pi/2$	121 16108	51472	612	0

Angular quantities are given in fractions of right angle.

The computation of the differences up to δ^6 requires the knowledge of three extra values (belonging to the second sheet of the Riemann surface), which are:

x	y	MODULUS	PHASE
−.3	2.3	1.96805 71778	+.575
−.2	2.2	1.92670 38010	.55
−.1	2.1	1.88881 87411	.525

Some numerical values, inherent to this case, are given in Table I, together with the first even horizontal differences.

7. The use of complex quantities can also be avoided in the computation of K,

using the formula, due to Gauss[6], which expresses K' in terms of the scale $(1, k')$ [or K in terms of the scale $(1, k)$]. We write preferably the formula in the form

$$\frac{\pi}{2}\frac{K}{K'} = \log\frac{4a_1}{c_0} + \tfrac{1}{2}\log\frac{a_2}{a_1} - \tfrac{1}{4}\log\frac{a_2}{a_3} - \tfrac{1}{8}\log\frac{a_3}{a_4} - \cdots \tag{4}$$

the terms being those of the scale $(1, k)$. Since the phase angle of a_1 is $-y/4$, and that of $c_0 = k'$ is $x/4 = (\pi - y)/4$, the imaginary part of $\log(4a_1/c_0)$ is constantly $-i\pi/4$ (if x is positive):

$$4a_1/c_0 = 2e^{-i\pi/4}[\tan\tfrac{1}{4}y]^{-1}.$$

The other fractions in (4) are all real, and, if the formula is written as above, all the logarithms are positive. The convergence of the series is extremely rapid: if $|x| \leq \pi$, the terms of the series explicitly written in (4) are sufficient for a computation to ten decimal figures.

The results of some computations, for the usual interval of x, are given in the column $R(K/K')$ of table IV. Knowing that the imaginary part of the ratio K/K' is constantly $\pm\frac{1}{2}i$, the modulus of the ratio is determined at once; the phase is deduced from the knowledge of the imaginary and real parts.

The corresponding values of K are given in table II, with some even differences. Since K has a point of logarithmic infinity at $k = 1$ $(x = \pi, y = 0)$ and its derivatives have poles of increasing order at this point, interpolation to a high number of places is not practical, unless x is small, and no interpolation is possible in the last interval.[7] A small subtabulation of the last interval, by direct computation, is given in table III. The values of K', also given in the table, were obtained as intermediate results.

8. The knowledge of the functions K and K' on the unit circle 0 automatically leads to the knowledge of the same functions on the unit circle 1, where they are simply interchanged. The determination of K on the line $R(z) = \frac{1}{2}$ is immediately possible by noting that

$$K(t/(t-1)) = \sqrt{1-t}\,K(t) = k'K(t)$$

and that, when $|t| = 1$, the real part of $z = t/(t-1)$ is $\frac{1}{2}$. Further, since $1 - z$ is conjugate to z on this line, the values of K' are simply conjugate to those of K.

The values of K are given in Table V, with the usual subtabulation for the last interval of x. As stated above, $K(z)$ and $K'(z)$ satisfy equation (2) if $z = t/(t-1)$, so that, when $t = -e^{ix}$, the real and imaginary parts of $K(z) = k'K(-e^{ix})$ give two independent periodic solutions of

$$8(1 + \cos x)\,d^2y/dx^2 + y = 0,$$

being respectively even and odd. Although the functions have singularities in the points $x = \pm\pi$, they vanish in these points, where their first derivatives

[6] King, page 13, form. (61).

[7] For the same reason, differencing does not afford a satisfactory check of the accuracy of the higher decimals. All the computed figures are given, so that the last digit must usually be regarded as uncertain.

have points of logarithmic infinity. The Fourier series in terms of x hence converge on the whole real axis, but cannot be differentiated term by term.

TABLE II

K for $k^2 = -e^{ix} = e^{-iy}$ *; or K' for* $k^2 = 1 + e^{ix} = 1 - e^{-iy}$

x	y	MODULUS	δ^2	δ^4	δ^6	PHASE
0	2ᴸ	1.31102 87772	+244 69860	+3 70708	+18544	0
.1ᴸ	1.9	1.31225 22702	246 55214	3 79980	19565	−.013 5717ᴸ
.2	1.8	1.31594 12846	252 20548	4 08817	22794	.027 1153
.3	1.7	1.32215 23538	261 94699	4 60448	28576	.040 6022
.4	1.6	1.33098 28929	276 29298	5 40655	39841	.054 0021
.5	1.5	1.34257 63618	296 04552	6 60703	53617	.067 2826
.6	1.4	1.35713 02859	322 40509	8 34368	85086	−.080 4082
.7	1.3	1.37490 82609	357 10834	10 93119	1 27190	.093 3388
.8	1.2	1.39625 73193	402 74278	14 79060	2 09973	.106 0277
.9	1.1	1.42163 38055	463 16782	20 74974	3 56908	.118 4204
1.0	1.0	1.45164 19699	544 34260	30 27796	6 44209	.130 4504
1.1	.9	1.48709 35603	655 79534	46 24827	12 44667	−.142 0350
1.2	.8	1.52910 31041	813 49635	74 66525	26 22065	.153 0676
1.3	.7	1.57924 76114	1045 86261	129 30288	61 90364	.163 4064
1.4	.6	1.63985 07448	1407 53175	245 84415	171 34455	.172 8545
1.5	.5	1.71452 91957	2015 04504	533 72997	632 13838	.181 1226
1.6	.4	1.80935 80970	3156 28830	1453 75417	3168 78833	−.187 7542
1.7	.3	1.93574 98813	5751 28573	5542 56670	∞	.191 9517
1.8	.2	2.11965 45229	13888 84986	∞		.192 0577
1.9	.1	2.44244 76632	∞			.183 4990
2	0	∞				0

$|K|$ is even with respect to x. The accuracy of the phases has been limited to seven decimals of quadrant, due to the limited approximation of the trigonometrical table used (J. Peters—*Siebenstellige Werte der trigonometrischen Funktionen*—Leipzig 1918). More accurate values can be deduced from the knowledge of the imaginary part of the ratio K/K' as $\pm\frac{1}{2}i$, and of the real part from Table IV.

9. For the computation of other elliptic, or modular, functions for the same complex values of the modulus, the knowledge of all intermediate terms of the a.g.m. would be useful. The knowledge of the whole scale, however, can be replaced, to some extent, by that of the *nome* q:

$$q = e^{-\pi K'/K}$$

or of the *complementary nome*

$$q' = e^{-\pi K/K'}.$$

q and q' are obviously connected by the simple relation

$$\log q = \pi^2/(\log q').$$

According to (4), the natural logarithm of q', for $k^2 = -e^{ix}$, (i.e., the logarithm of q, for $k^2 = 1 + e^{ix}$) has the imaginary part $\pm i\pi/2$; that is, q' is purely imaginary. Its absolute value can be computed by using decimal logarithms

TABLE III

$k^2 = -e^{ix} = e^{-iy}$; K and K' in the vicinity of $k^2 = 1$

x	y	K		K'	
		Modulus	Phase	Modulus	Phase
1.9°	.1°	2.44244 76632	−.183 4990°	1.57140 21323	+.025°
1.91	.09	2.49223 36486	.181 6678	1.57128 69964	.0225
1.92	.08	2.54805 09501	.179 5273	1.57118 39930	.02
1.93	.07	2.61151 92118	.177 0078	1.57109 31179	.0175
1.94	.06	2.68501 23477	.174 0093	1.57101 43674	.015
1.95	.05	2.77221 44086	.170 3837	1.57094 77380	.0125
1.96	.04	2.87930 35219	.165 8931	1.57089 32269	.01
1.97	.03	3.01787 57239	.160 1150	1.57085 08318	.0075
1.98	.02	3.21400 30414	.152 0520	1.57082 05509	.0050
1.99	.01	3.55104 52306	.136 0733	1.57080 23827	.0025
2	0	∞	0	1.57079 63268	0

only; since, if the left side of (4) is written $-\frac{1}{2}\log q'$, all terms of the formula are logarithmic. If $M = \log_{10} e$, we simply have:

$$-\log_{10} |q'| = M\pi R(K/K').$$

q is immediately deduced from

$$\log_{10} q = M^2\pi^2/(\log_{10} q').$$

The period-ratio of the epllitic functions, related to the same values of the modulus: $\tau = iK/K'$, is deduced at once from Table IV. It has the real part $\pm\frac{1}{2}$.

10. The integrals of the second kind can also be expressed in terms of the a.g.m. scale, by means of the relation[8]

$$(K - E)/K = \tfrac{1}{2}\{c_0^2 + 2c_1^2 + 4c_2^2 + \cdots\}.$$

[8] King, page 8, form. (27).

TABLE IV

Functions of the modulus, given $k^2 = -e^{ix}$

The coefficient of i in K/K' is $\pm\frac{1}{2}$, according to the sign of x, and to the assumed determination for $k' = \sqrt{1-k^2}$. Writing as usual $q = e^{-\pi K'/K}$, $q' = e^{-\pi K/K'}$, q' is the function q relative to a $k^2 = 1 + e^{ix}$. q' is purely imaginary; for a given k^2, we have $log\ q = \pi^2/log\ q'$ (natural logarithms). Also $log_{10} | q' | = -M\pi R(K/K')$, with $M = log_{10} e$.

x	k′ Modulus	k′ Phase	$R\left(\frac{K}{K'}\right)$	$\left\|\frac{K}{K'}\right\|$	$-log_{10}\|q'\|$
0	1.41421 35624	0	.5	.70710 67812	.68218 81769
.1	1.41203 21056	+.025	.51828 17061	.72014 99336	.70713 13045
.2	1.40548 09430	.05	.53730 42187	.73395 90066	.73308 51708
.3	1.39453 92934	.075	.55717 70028	.74862 95562	.76019 91275
.4	1.37917 11397	.1	.57802 68303	.76427 41763	.78864 61391
.5	1.35932 30171	.125	.60000 26895	.78102 70337	.81862 94817
.6	1.33492 06150	.15	.62328 22895	.79904 99436	.85039 16175
.7	1.30586 38247	.175	.64808 09825	.81854 07503	.88422 63679
.8	1.27201 96495	.2	.67466 40429	.83974 49439	.92049 56669
.9	1.23321 20382	.225	.70336 44575	.86297 25141	.95965 38340
1.0	1.18920 71150	.25	.73460 90158	.88862 27580	1.00228 31705
1.1	1.13969 12286	.275	.76895 82186	.91722 22969	1.04914 84105
1.2	1.08423 72916	.3	.80716 97428	.94948 56469	1.10128 33106
1.3	1.02225 10109	.325	.85030 44725	.98641 65935	1.16013 53159
1.4	.95288 03700	.35	.89991 47891	1.02948 85272	1.22782 24587
1.5	.87485 24817	.375	.95840 60570	1.08099 12905	1.30762 65615
1.6	.78615 13777	.4	1.02980 65911	1.14477 14248	1.40504 37619
1.7	.68329 40273	.425	1.12166 66723	1.22806 19381	1.53037 54846
1.8	.55918 26861	.45	1.25112 22839	1.34733 32807	1.70700 16599
1.9	.39612 90085	.475	1.47169 38441	1.55431 10277	2.00794 42811
1.91	.37583 76934	.4775	1.50523 89136	1.58610 97651	2.05371 23805
1.92	.35437 41511	.48	1.54273 73747	1.62173 93771	2.10487 43941
1.93	.33151 22167	.4825	1.58524 78978	1.66223 06992	2.16287 47466
1.94	.30694 12019	.485	1.63432 08787	1.70909 47120	2.22982 87614
1.95	.28021 35464	.4875	1.69236 01327	1.76467 64063	2.30901 61472
1.96	.25064 22115	.49	1.76339 26132	1.83290 84834	2.40593 11839
1.97	.21707 03333	.4925	1.85496 75251	1.92117 26937	2.53087 38282
1.98	.17724 17406	.495	1.98403 31226	2.04606 63312	2.70696 78776
1.99	.12533 07695	.4975	2.20466 99478	2.26065 68910	3.00799 95446
2	0	.5	∞	∞	∞

TABLE V

The function $k'K$ for $k^2 = -e^{ix}$, or $K(z)$ for $z = \frac{1}{2} + iv$ ($v = \frac{1}{2} \tan \frac{1}{2}x$)

x	v	MODULUS	PHASE	REAL PART	IMAG. PART
0	0	1.85407 46773	0	1.854 0747	0
.1	.039 3509	1.85294 23362	.011 4286	1.852 6436	.033 2614
.2	.079 1922	1.84953 03976	.022 8847	1.848 3356	.066 4710
.3	.120 0394	1.84379 34092	.034 3978	1.841 1026	.097 7314
.4	.162 4598	1.83565 31933	.045 9979	1.830 8637	.132 5167
.5	.207 1068	1.82499 49508	.057 7174	1.817 4995	.165 2318
.6	.254 7627	1.81166 11959	.069 5918	1.800 8476	.197 6448
.7	.306 4004	1.79544 29602	.081 6612	1.780 6921	.229 6761
.8	.363 2713	1.77606 67459	.093 9723	1.756 7524	.261 2166
.9	.427 0403	1.75317 59229	.106 5796	1.728 6630	.292 1385
1.0	.5	1.72630 29590	.119 5496	1.695 9539	.322 2771
1.1	.585 4248	1.69482 74868	.132 9650	1.657 9950	.351 4152
1.2	.688 1910	1.65791 06082	.146 9324	1.613 9486	.379 2592
1.3	.815 9258	1.61438 74672	.161 5936	1.562 6587	.405 3946
1.4	.981 3053	1.56258 15844	.177 1455	1.502 4767	.429 2138
1.5	1.207 1068	1.49996 01218	.193 8774	1.430 9386	.449 7725
1.6	1.538 8418	1.42242 93607	.212 2458	1.344 1058	.465 4943
1.7	2.082 6499	1.32268 63322	.233 0483	1.235 0461	.473 4556
1.8	3.156 8758	1.18527 41097	.257 9423	1.089 6646	.467 2115
1.9	6.353 1024	.96752 43711	.291 5010	.867 8576	.427 6993
1.91	7.061 7681	.93667 53459	.295 8322	.837 3499	.419 7688
1.92	7.947 2963	.90296 33929	.300 4727	.804 2417	.410 5340
1.93	9.085 4034	.86575 05622	.305 4922	.768 4122	.399 6827
1.94	10.602 4744	.82414 09171	.310 9907	.727 7464	.386 7730
1.95	12.725 8498	.77681 20303	.317 1163	.682 4137	.371 1448
1.96	15.910 2580	.72167 50017	.324 1069	.630 1526	.351 7420
1.97	21.216 7320	.65509 22402	.332 3850	.567 8138	.326 7007
1.98	31.828 3706	.55965 54934	.342 9480	.488 9784	.292 2457
1.99	63.660 6681	.44505 52316	.361 4267	.375 2372	.239 3140
2	∞	0	.5	0	0

If the terms c_n are those relative to the mean $(1, k)$, the formula defines E' in terms of K'. In the formula, $c_0^2 = k'^2 = 1 + e^{ix}$ has the phase $\frac{1}{2}x = \frac{1}{2}(\pi - y)$; the terms $c_n^2 = a_n^2 - b_n^2$ $(n \geq 1)$ all have the phase $-\frac{1}{2}y$. In the expression

$$e^{iy/2}(K' - E')/K' = \tfrac{1}{2}\, e^{iy/2}\{c_0^2 + 2c_1^2 + 4c_2^2 + \cdots\}$$

the right side is, therefore, a complex number having $|\, \tfrac{1}{2}c_0^2\,| = \cos \tfrac{1}{2}\, x$ as coefficient of i, and the series

$$-\,|\,c_1^2\,| + 2\,|\,c_2^2\,| + 4\,|\,c_3^2\,| + \cdots$$

TABLE VI

Real and imaginary parts of the expression $e^{iy/2}\{1 - E'/K'\}$ *for* $k^2 = -e^{ix} = e^{-iy}$

x	y	REAL	IMAGINARY
0	2	−.45694 65815	1.
.1+	1.9+	−.42537 34309	.99691 73337
.2	1.8	−.39301 30347	.98768 83406
.3	1.7	−.36019 09513	.97236 99204
.4	1.6	−.32722 86982	.95105 65163
.5	1.5	−.29444 19428	.92387 95325
.6	1.4	−.26213 86775	.89100 65242
.7	1.3	−.23061 73614	.85264 01644
.8	1.2	−.20016 51138	.80901 69944
.9	1.1	−.17105 59141	.76040 59656
1.0	1.0	−.14354 88792	.70710 67812
1.1	.9	−.11788 66192	.64944 80483
1.2	.8	−.09429 36760	.58778 52523
1.3	.7	−.07297 50811	.52249 85647
1.4	.6	−.05411 50177	.45399 04997
1.5	.5	−.03787 56290	.38268 34324
1.6	.4	−.02439 59532	.30901 69944
1.7	.3	−.01379 10165	.23344 53639
1.8	.2	−.00615 10786	.15643 44650
1.9	.1	−.00154 13332	.07845 90957
2	0	0	0

as real part.[9] In spite of the increase of the values of the numerical coefficients, the series converges with the usual rapidity: the written terms already suffice for a computation to nine decimals in the whole range $0 \leq |\, x \,| \leq \pi$.

As soon as E' has been determined, E is also deduced immediately from

[9] Since $c_1^2 = a_1^2 - b_1^2$ is negative, and much larger in modulus than the remainder of the series, the series itself is negative.

Legendre's relation

$$E'K + EK' - KK' = \pi/2$$

The values of the ratio $e^{\pi/2}[1 - E'/K']$ for the usual values of k^2 are given in table VI. The deduction of the explicit values of E' (and E) from the given values is a matter of elementary computation.[10]

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[10] All the computed figures are given, so that the last one is sometimes unsafe.

CONTRIBUTION TO THE HEREDITY THEORY OF MULTIVALENTS*

By Hilda Geiringer

Introduction. It is generally agreed today that Mendel's theory of Heredity presents one of the really great contributions to Science. Based on observations, it is a highly original, constructive theory by which great domains of previously more or less unintelligible experience is unified and clarified. These ideas mark the beginning of mathematical biology, of genetics as a science. Mathematically the theory belongs to probability theory; it is one aim of the present paper (and of other related ones) to understand and describe in terms of probability theory certain genetical situations which may be considered as generalizations of Mendel's original problems.

Mendel observed that certain hereditary attributes, called now (Mendelian) *characters* are distributed according to a specific law of probability. The color of the flower of peas is such an attribute. That means, mathematically speaking, it is to be considered as a random variable which, in this case, may take on $r = 2$ possible "values", red or white ("1" or "2"). In case of $r \geqq 2$ "*alleles*" (e.g. r different colors) the random variable, x, takes on the values $1, 2, \cdots r$. The genetic constitution of the organism, the plant in our example, is specified not by one but by *two* values x and y of this random variable, by two *genes*, which represent the plants maternal and paternal heritage respectively, with respect to this color. Assuming distinguishable and non overlapping *generations* there exists in the n^{th} generation a symmetric *distribution of genotypes* $w^{(n)}(x, y)$ $(x = 1, \cdots r, y = 1, \cdots r)$, $(n = 0, 1, 2, \cdots)$ and $w^{(n)}(x, y) = w^{(n)}(y, x)$. (We assume that this distribution be the same for males and females.) In the process of procreation the parent, of constitution (x, y), transmits to the offspring one of its two genes, the *gamete*, (sperm or egg) the other gene coming from the other parent. This happens again according to a probability distribution and it is assumed that there is equal probability for the transmittance of a parent's two genes. Thus this probability distribution which we shall name *segregation distribution* (s.d.) is here a simple "alternative" with the probabilities $p = q = \frac{1}{2}$ for the transmittance of the maternal or paternal gene respectively. This s.d. will not be so simple in more general situations but we shall hold to three of the assumptions made here, that the s.d. does not depend on n (is the same through the generations), is the same for males and for females and is independent of the parent's genotype.

If we consider *random breeding* (panmixia), (i.e. no selection whatsoever, no difference in vitality between the different types, etc. but simple multiplication of the probabilities of the available types,) the third important distribution, the *distribution of gametes* can easily be found in terms of the two others. Denote by $p^{(n)}(x)$ the probability of the gamete x in the n^{th} generation, where $x = 1, \cdots r$,

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and $\sum_x p^{(n)}(x) = 1$. We have e.g.:

$$p^{(n)}(1) = w^{(n)}(1, 1)(p + q) + w^{(n)}(1, 2)p + w^{(n)}(2, 1)q + \cdots$$

and since $p = q = \frac{1}{2}$ and $w^{(n)}(x, y) = w^{(n)}(y, x)$ this gives simply:

$$p^{(n)}(x) = \sum_y w^{(n)}(x, y).$$

(Here the s.d. does not appear since it contains no recognizable parameters.) As a new genotype is formed by the fusion of the two gametes the distribution of genotypes *in the following generation* is simply: $w^{(n+1)}(x, y) = p^{(n)}(x) \cdot p^{(n)}(y)$, and that finishes the cycle. From these simple formulae we deduce at once G. H. Hardy's [6] now classical result. In fact computing $p^{(n+1)}(x)$, we see,—in one line—that:

$$p^{(n+1)}(x) = \sum_y^{1 \cdots r} w^{(n+1)}(x, y) = \sum_y^{1 \cdots r} p^{(n)}(x) \cdot p^{(n)}(y) = p^{(n)}(x) \qquad (n = 0, 1, 2, \cdots).$$

In words: In case of a "diploid" organism and one character the distribution of gametes remains constant throughout under random breeding, while the distribution of genotypes remains constant from the first filial generation on:

$$p^{(n)}(x) = p^{(0)}(x), \qquad w^{(n)}(x, y) = w^{(1)}(x, y), \qquad (n = 1, 2 \cdots).$$

We see that in this simplest case no "recurrence problem" arises since, after *one* good mixing, the characteristic distributions do not change any more. This is not a typical result. Equilibrium is reached, in general, as $n \to \infty$ only. However the main ideas and concepts introduced in the simplest case remain valid in much more general situations: The genetic constitution of the organism is defined by the genetic constitution of the two gametes, which represent maternal and paternal heritage; each gamete is however in general no longer a single gene but a more or less complicated set of genes. In the notation $w^{(n)}(x, y)$ for the d. of genotypes, x and y now stand each for a gamete. The kinds of gametes which such an organism may, in turn, produce correspond to the combinations of the total genetic material it has inherited. This segregation happens again according to a probability law which we call again the s.d. It is one of the primary tasks of the theory to define this s.d. under general conditions, as e.g. stated in a recent paper of R. A. Fisher [2b]: "The laws of inheritance obtained by genetic studies are the rules whereby given the constitution of an organism the kinds of gametes it can produce and their relative frequencies can be predicted." By means of this s.d. the d. of gametes, $p^{(n)}(z)$ follows, in principle, from the d. of genotypes $w^{(n)}(x, y)$ and $w^{(n+1)}(x, y)$ is then determined by $p^{(n)}(z)$. A complete cycle of inheritance is thus described. It seems easy to enumerate a few problems which present themselves to the mathematician in this connection:

a) The possible genotypes corresponding to a certain biological situation are to be completely and simply enumerated under consideration of particular

biological conditions which may influence the number of possible types. b) The s.d. adapted to the biological problem is to be defined.—Theoretically minded geneticists have of course always been concerned with these problems although the problems and answers may sound very differently in a different "language". While the basic facts and suggestions must necessarily come from the biologist, from his interpretation of observations and his creative imagination, the mathematician can help to clarify and simplify the concepts. c) The d. of gametes $p^{(n)}(z)$ is to be derived from the d. of genotypes $w^{(n)}(x, y)$ by means of the s.d. Then $w^{(n+1)}(x, y)$ follows. *Recurrence relations* are desired between $p^{(n)}(z)$ and $p^{(n+1)}(z)$. d) It is desirable to *integrate* these recurrence equations (in general a system of quadratic difference equations), i.e. to determine $p^{(n)}(z)$ in terms of "n" and of the initial values $w^{(0)}(x, y)$, . . (and of course of parameters introduced in the definition of the s.d. or otherwise). e) We want to know whether and if so under what conditions an *equilibrium status* is reached for $p^{(n)}(z)$, and consequently for $w^{(n)}(x, y)$. In connection with this question we have to derive from the recurrence formulae the limit behaviour of these distributions as $n \to \infty$.—The last problems, c), d), e) seem to be of a fairly theoretical character. They are however of practical interest as well, as their results may be checked by means of appropriate observations and thus furnish a check for the basic biological assumptions of the genetic theory.

Let us return to the consideration of the historical development. A first generalization consists in the study of *several* characters, e.g. the color of the flower and the form of the seeds of a plant. Mendel assumed that *independent assortment* prevailed, i.e. that all possible combinations of gametes are equally probable. This, of course implies a very special assumption regarding the s.d. This problem of m mutually independent characters has been completely solved by H. Tietze [13] in a very interesting paper. Under these assumptions the s.d. does still not contain a single parameter.

This conception of the problem was shaken through the observation of "linkage" in the famous investigations with drosophyla melanogaster, connected with the names of Morgan and his collaborators. Obviously not all possible gametes appear with equal frequency. If $m = 2$ *one* parameter is to be introduced, the crossover value. The theoretical study of this case was accomplished by H. S. Jennings [7], R. B. Robbins [11], W. Weinberg [14]. In the general case there are $m(m-1)/2$ crossover values and it is easily seen that a priori, i.e., without the use of additional biological hypotheses this number of parameters is not sufficient for the description of general linkage. In fact, *in case of m factors* a genotype is characterized by two m-dimensional "vectors", x_i and y_i $(i = 1, \cdots m)$. In the formation of the gamete a new set of m elements is composed in such a way that for each of the m subscripts either the x-value or the y-value is chosen; hence 2^m possibilities. For obvious symmetry reasons the number of parameters that may enter the s.d. reduces to $M = 2^{m-1} - 1$ a number which is greater than $m(m-1)/2$ as $m \geqq 4$. Under the assumption of *general linkage between m characters* this author has formulated and solved the aforementioned problems [4c], [4d]. (What I name now segregation distribution

was called "linkage distribution" in these papers.) The problem belongs mathematically to the probability theory of "arbitrarily linked events" [4a], [4b] which has been studied particularly in France.

A second extension of Mendel's original problem has come with the discovery of *polysomic inheritance*. This is the problem that will be dealt with in the present paper.

Polysomic inheritance, as observed and interpreted by Gregory, Muller [10], Haldane [5], R. A. Fisher [2], Mather [8] and others presents new problems to the geneticist. In a basic paper, [5], Haldane has studied the theoretical genetics of so called *autopolyploids* and has found remarkable results that will be quoted in detail. To K. Mather and R. A. Fisher [2a], [2b], [3], [8] other important theoretical investigations are due. They deal however, particularly, with the consequences of "double reduction", a problem which we shall study in a separate paper, dealing with "chromatid segregation", in contrast to "chromosome segregation" considered in the present paper. We also shall restrict ourselves here to the consideration of *one* character, the problem of linkage of polyploids being reserved to another publication. Random mating will be assumed throughout and "orthopolyploids" (see sec. 1).

The main feature of a polyploid organism is, in case of one character, that each of the two gametes consists not of one but of $s \geqq 1$ genes. We thus consider *2s-valent* organisms instead of the *bivalents* discussed so far. A genotype is then described as follows: The 2s-valent organism possesses two sets of s genes, each gene being given by one of r numbers $a_1, \cdots a_r$, the r "alleles" (e.g. r genotypically different colors); r may be less than, greater than or equal to s. These two sets of numbers, each number equal to one of the a_i describe the genotype of the organism. The *distribution of genotypes* is a discrete probability distribution in 2s variables. There are however less than r^{2s} distinct genotypes, because of characteristic biologically determined restrictions.

The results of observations suggest (in analogy to the theory of diploids), as the next step the following formulation: In the formation of a new individual each parent transmits to the offspring one set of s genes. The selection of these s transmitted genes happens according to a probability law which we assume to be the same for males and females and the same through the generations. Out of 2s numbers $S = \binom{2s}{s} = \frac{(2s)!}{s!\,s!}$ sets can be selected (compare however later remarks about "chromatid segregation"). These S cases are more or less tacitly assumed as equally probable by Muller, who was the first to advance this, Haldane, and others, i.e. "random chromosome segregation" is considered. This assumption will not be made here. We introduce a s.d. where the S probabilities are not a priori equal to each other. While attempting to keep this distribution sufficiently general we must avoid to introduce too many, biologically not meaningful parameters. The s.d. plays an essential role in our proofs, its values acting as "separators" between groups of expressions. At the same time the results become more general. Certain observational facts make us believe that our s.d. yields an adequate representation of the biological situation.

An individual has received s genes from its mother, s from its father. It is basic in the author's conception to characterize a type mathematically not by $2s$ numbers but by s "maternal" and s "paternal" values, i.e. by two sets of s numbers each. Studying the results of observations one is lead to think that the proportion of maternal and paternal genes within the s transmitted genes (i.e. within the gamete) is of biological importance. We thus make the following definition: Call $l_\alpha = \binom{s}{\alpha}\lambda_\alpha$, $(\alpha = 0, 1, \cdots s)$, the probability that a specified set of α maternal genes be transmitted. Assume $l_\alpha = l_{s-\alpha}$ (symmetry of paternal and maternal heritage). Since a set of α specified maternal genes may be combined in the gamete with any $(s - \alpha)$ paternal genes we have

$$\sum_{\alpha=0}^{s}\binom{s}{\alpha}l_\alpha = \sum_{\alpha=0}^{s}\binom{s}{\alpha}^2\lambda_\alpha = 1.$$

Thus if $s = 2\mu$ or $2\mu + 1$ respectively just μ parameters are introduced which may, in particular cases, be assumed to be in part or all equal to each other. For $\lambda_\alpha = 1/S$ we have random chromosome segregation (Haldane, Muller).

In sections 2 and 3 a general *recurrence formula* is derived which permits to compute the d. of genotypes from generation to generation. This law is of a remarkably clear and simple structure thanks to the above concepts and to the use of a certain type of "marginal distributions" (sec. 2), a concept well known in probability calculus. (Explicit computation of recurrence relations by enumerating all possibilities, (as tried by some authors) seems hopeless. (see page 260)). (In our recurrence relations (19), (64) the values of the s.d. play a decisive role. Since the recurrence relations formulate biological facts we conclude once more that our s.d. may be biologically meaningful. From the recurrence relations, by means of auxiliary results, (sec. 4) the *limit theorem*, is derived as, n, the number of distinct generations, increases without limit. Haldane has shown that a certain distribution, in general identical with our limit distribution, reproduces itself under random segregation. This however left the biological and mathematical question open whether, and if so under what conditions, such an equilibrium is actually reached. Next, for every s, a procedure for *solving the recurrence equations* is developed. They form a system of quadratic difference equations with constant coefficients, these coefficients being the values of the s.d. (sec. 6). In particular the *rate of convergence* towards the solution is computed for any s. (Haldane gives the explicit solution for $s = 2$ and the rate of approach towards equilibrium for $s = 3$.)

All our results are established under the assumption of a "chromosome segregation". In diploids the segregation proportions are the same no matter whether the two chromosomes or the four chromatids are considered. The segregation problem for polyploids is a different one in the two cases. The more complex possibilities of "chromatid segregation" will be considered in a separate paper under assumption of a segregation d. general enough to include as a particular case the chromosome segregation introduced in the present paper as well as as-

sumptions considered by Haldane, Fisher, and Mather. The author's approach is there again to establish recurrence formulae (using the s.d.), to study their integration and the limit behaviour of the characteristic distributions.

In most biomathematical papers *two* alleles only are considered. (This includes all papers quoted so far.) The concepts and results become however almost clearer if multiple alleles are admitted. Nevertheless it was chosen to start in this paper with the investigation of two alleles since this is easier for the biologist, (the two alleles often corresponding to "dominant" and "recessive" properties) while for the mathematician it may make the biological situation more tangible. In sections 7 and 8 only, the case of a $2s$-ploid and r alleles is considered. The limit theorem takes the following elegant form: *The gametic proportions corresponding to each allele remain constant through the generations. The joint distribution of gametes converges towards a limit where these alleles are independently distributed if and only if* $l_0 < \frac{1}{4}$.

This limit theorem as well as the main theorem for linkage [4d] appear as particular cases of a general limit theorem for polyploids with m characters and any number of alleles.

1. The distribution of genotypes and the segregation distribution. This paper is concerned with "autopolyploids" and among them with "orthopolyploids." A genotype (zygote) possesses $2s$ chromosomes ($s \geqq 1$). We shall use the term zygote in contrast to gamete. Of these $2s$ chromosomes one set consisting of s has been received from its mother, the other from its father. Separating these two sets by a semicolon the type of a zygote if *one character* and *two alleles* A and a are considered may be denoted by

$$(1) \qquad (A^x a^{s-x}; A^y a^{s-y}) \qquad (x = 0, 1, \cdots s; \quad y = 0, 1, \cdots s).$$

Here $A^x a^{s-x}$ indicates that there are x A-values and $(s - x)$ a-values and that we consider no difference of order, i.e. no difference between $AAaaA$ and $aaAAA$. There are obviously a priori $N' = 2^{2s}$ possible types since there are $\binom{s}{x}$ arrangements of $A^x a^{s-x}$ and $\sum_{x=0}^{s} \binom{s}{x} = 2^s$. Among these there are however not more than $N = (s + 1)^2$ different types since before, as well as after the semicolon each of the $(s + 1)$ combination $A^s, A^{s-1}a, \cdots a^s$ can be written. If we further assume *symmetry*, so that the letters before and after the semicolon (the paternal and maternal heritage) play exactly the same role (e.g. $(A^2a^2; a^5) = (a^5; A^2a^2)$) then there remain $N_1 = (s + 1)(s + 2)/2$ different types only. Finally if, as e.g. in Haldane's work, the distinction between paternal and maternal heritage is not emphasized there are just $N_2 = (2s + 1)$ types left: $A^{2s}, A^{2s-1}a, \cdots a^{2s}$. For the diploid, $s = 1$, we have $N = 4$ and $N_1 = N_2 = 3$. For the tetraploid ($s=2$): $N = 9, N_1 = 6, N_2 = 5$, for the hexaploid ($s = 3$): the three figures are 16, 10, and 7. (To the type A^4a^2 in Haldane's conception correspond in the other conception the types $(A^3; Aa^2)$ and $(A^2a; A^2a)$, etc.) We shall *adopt here the*

assumption leading to the N_1 different types. The idea to distinguish throughout the maternally derived genes from the paternal ones proved essential in the author's paper on linkage.

If we call $w(x; y) \equiv w(A^x a^{s-x}; A^y a^{s-y})$ the probability of a *particular* type $(A^x a^{s-x}; A^y a^{s-y})$ then, for fixed x and y, this type presents itself $\binom{s}{x}\binom{s}{y}$ times. Hence:

$$\sum_{x=0}^{s}\sum_{y=0}^{s}\binom{s}{x}\binom{s}{y} w(x; y) = 1 \tag{2}$$

and we assume that

$$w(x; y) = w(y; x). \tag{3}$$

Now consider the *gametes.* We assume that all gametes containing z A-values and $(s - z)$ a-values represent the same type. Hence such a type can be denoted by $(A^z a^{s-z})$ or simply by the number (z). There are 2^s gametes, but $(s + 1)$ different types only and if we denote by $p(A^z a^{s-z})$ the probability of such a particular type we have

$$\sum_{z=0}^{s}\binom{s}{z} p(z) \equiv \sum_{z=0}^{s}\binom{s}{z} p(A^z a^{s-z}) = 1. \tag{4}$$

From the point of view of systematic probability theory the d. $p(A^z a^{s-z})$ is an "s-dimensional alternative", since it is a function of s chance variables each of which can have two values only, a and A, or 0 and 1. But there is a characteristic symmetry-restriction in the sense that of the 2^s probabilities all these which have the same *number* z of "ones" (or the *same sum* z) have equal values. Next consider the segregation-distribution. In the formation of a new individual a zygote transmits s genes the other set of s-values coming from the other parent. Such a set of s-genes can be selected from the $2s$ parental ones in $\binom{2s}{s} = S$ ways if no restriction is assumed. This number S can also be interpreted in the following way: A gamete may have received α paternal and $(s - \alpha)$ maternal genes and since $\alpha = 0, 1, \cdots s$ there are obviously

$$\binom{s}{0}\binom{s}{s} + \binom{s}{1}\binom{s}{s-1} + \binom{s}{2}\binom{s}{s-2} + \cdots + \binom{s}{s}\binom{s}{0} = \binom{2s}{s} = S \tag{5}$$

possibilities. For $s = 1, 2, 3, 4, 5, \cdots$ these numbers are: $S = 2, 6, 20, 70, 252, \cdots$ etc.

In Haldane's and Muller's contributions it is silently assumed that each of these s possibilities has equal probability $1/S$. If this is so the three types of gametes transmitted by a zygote of type $(A^2; a^2)$ should appear in the proportions 1:4:1. The figures reported in several papers do not show these proportions with significant accuracy. From a theoretical point of view the uniform dis-

tribution just mentioned seems a very special assumption. It may seem useful from the biological point of view to advance a more general theory not based on such an a-priori restriction. It could well be imagined that there is some "linkage" which acts e.g. as a tendency of the maternal (paternal) genes to stay more or less closely together. If this is so then for an individual of type $(A^5; a^5)$ e.g. the probability to transmit A^5 may be different from that of forming a "mixed" gamete, say of constitution A^2a^3, where two maternal genes are combined with three paternal ones. The same holds of course for a homozygote $(A^5; A^5)$ where the only possible gamete of type A^5 may contain 0 to 5 maternal values.

In the author's opinion it makes sense, from a biological point of view, not to assume a priori a uniform distribution for the $S = \binom{2s}{s}$ possibilities. But apart from this it is, mathematically, a most useful artifice to introduce a more general segregation-distribution (s.d.) for these S probabilities. It will help in collecting and separating various groups of probabilities in an appropriate way and thus enable us to derive clear recurrence formulae. The parameters thus introduced may be fitted to and checked with the observations. Of course, uniform distribution will appear as the simplest particular case.

This s.d. contains $S = \binom{2s}{s}$ probabilities. The following biologically meaningful and very simple assumptions will be made in order to achieve sufficient generality but to avoid unnecessary complications. *Call* $\lambda_\alpha (\alpha = 0, 1, s)$ *the probability that a certain fixed set of* s *genes be transmitted which contains a fixed group of* α *maternal and* $(s - \alpha)$ *paternal genes.* There are for each α, $\binom{s}{\alpha}^2$ probabilities λ_α because α genes can be chosen from the maternal set in $\binom{s}{\alpha}$ ways and $(s - \alpha)$ from the paternal set in $\binom{s}{s-\alpha} = \binom{s}{\alpha}$ ways. We assume that these $\binom{s}{\alpha}^2$ probabilities are all *equal to each other.* Furthermore taking into account the equivalence of the paternal and maternal set in our heredity assumptions we state that

$$\lambda_\alpha = \lambda_{s-\alpha} \qquad (\alpha = 0, 1, \cdots s). \tag{6}$$

We then have

$$\sum_{\alpha=0}^{s} \binom{s}{\alpha}^2 \lambda_\alpha = 1 \tag{7}$$

e.g. for $s = 2, 3, 4, 5, \cdots$

$s = 2$: $\lambda_0 + 4\lambda_1 + \lambda_2 = 1$ or $2\lambda_0 + 4\lambda_1 = 1$

$s = 3$: $\lambda_0 + 9\lambda_1 + 9\lambda_2 + \lambda_3 = 1$ or $2\lambda_0 + 18\lambda_1 = 1$

$s = 4$: $\lambda_0 + 16\lambda_1 + 36\lambda_2 + 16\lambda_3 + \lambda_4 = 1$ or $2\lambda_0 + 32\lambda_1 + 36\lambda_2 = 1$

$s = 5$: $\lambda_0 + 25\lambda_1 + 100\lambda_2 + 100\lambda_3 + \cdots = 1$ or $2\lambda_0 + 50\lambda_1 + 200\lambda_2 = 1$

In the following it will be convenient to introduce also

$$l_\alpha = \binom{s}{\alpha}\lambda_\alpha \tag{8}$$

the probability that a fixed set of α maternal chromosomes be transmitted. (7) then takes the form:

$$\sum_{\alpha=0}^{s}\binom{s}{\alpha}l_\alpha = 1 \tag{7'}$$

which can be written symbolically as

$$(1 + l)^s = 1 \qquad \text{with } l^i \equiv l_i \qquad (i = 0, 1, \cdots s). \tag{7''}$$

We get uniform distribution by putting each $\lambda_\alpha = 1/S$ or $l_\alpha = \binom{s}{\alpha}\cdot 1/S$. For $s = 2$ and $s = 3$ our assumption introduces just *one* parameter, l_0, for $s = 4$ and 5 we have two parameters and, in general ν parameters if $s = 2\nu$ or $2\nu + 1$. We may, of course, consider particular assumptions with less parameters, e.g. l_0 only different from the other values.

Always, $(l_0 + l_s)$ is the probability that only paternal or only maternal values are transmitted and $\binom{s}{\alpha}(l_\alpha + l_{s-\alpha}) = 2\binom{s}{\alpha}l_\alpha$ is the probability that among the s transmitted values there are α from one and $(s - \alpha)$ from the other set.

2. The distribution of gametes and its marginal distributions. Recurrence formulae for $s = 2$. Let us consider the distribution of gametes more closely. Denote by $\binom{s}{z}p^{(n)}(A^z a^{s-z})$ the probability that a gamete belonging to the n^{th} generation be of type $A^z a^{s-z}$, while $p^{(n)}(A^z a^{s-z})$ is the probability of a gamete of type $A^z a^{s-z}$ which possesses the z A's in a particular arrangement, e.g. at the first z places there are A's only.

We shall need certain "marginal distributions" of these $p^{(n)} = p$. Consider e.g. for $s = 3$: $p(A^3) + p(A^2a)$ where $p(A^2a)$ is the probability that the first two values are A's, the third one an "a". Then, $p(A^3) + p(A^2a)$ is the probability that the first two genes be A's. That probability could be denoted by $p_{12}(A^2)$ or more briefly by $p(A^2)$ since, according to our assumptions $p_{23}(A^2) = p(AAA) + p(aAA)$ has obviously the same value as $p_{12}(A^2)$ and as $p(AAA) + p(AaA) = p_{13}(A^2)$. More generally we write for a marginal distribution of order $(s - 1)$:

$$p(A^z a^{s-z}) + p(A^{z-1}a^{s-z+1}) = p(A^{s-1}a^{s-z}) \tag{9}$$

and

$$\sum_{x=0}^{s-1}\binom{s-1}{x}p(A^x a^{s-1-x}) = 1. \tag{10}$$

In general we derive from the given distribution $p(A^s a^{\sigma-s})$ the marginal distributions of order $\sigma = s-1, s-2, \cdots 2, 1$. They may be denoted in a symbolical way by

$$p(A^s a^{\sigma-s}) = p(A^s a^{\sigma-s}[A+a]^{s-\sigma}) \tag{11}$$

since

$$p(A^s a^{\sigma-s}) = \sum_{x=0}^{s-\sigma} \binom{s-\sigma}{x} p(A^{s-\sigma+x} a^{\sigma-s+x}) \tag{11'}$$

$$= \sum_{y=0}^{s-\sigma} \binom{s-\sigma}{y} p(A^{s+y} a^{\sigma-s-y})$$

and for these marginal distributions

$$\sum_{z=0}^{\sigma} \binom{\sigma}{z} p(A^s a^{\sigma-s}) = 1. \tag{11''}$$

As an example consider for $s = 5$:

$$\begin{aligned} p(Aa) &= p(A^4a) + 3p(A^3a^2) + 3p(A^2a^3) + p(Aa^4) \\ &= p(A^3a) + 2p(A^2a^2) + p(Aa^3) \\ &= p(A^2a) + p(Aa^2). \end{aligned}$$

It should be clearly understood that by "marginal distributions" we mean here the same as otherwise in probability calculus. The consideration of the "distribution of gametes" as a function of s chance variables is one of the essential points in our conception of the problem. But once this has been accepted the problem becomes amenable to probability calculus. In particular the use of the "marginal distribution" will bring out the simple regularity expressed in our recurrence formulae which, in a certain sense, constitute our main result, since the limit theorem, etc. depends on them. As a consequence of the particular symmetry properties of our distribution of gametes it follows that all $p_i(x_i)$ are equal to each other, all $p_{ij}(x_i x_j)$ equal to each other, etc. and of course, the "symbolical" rule (11) is likewise a consequence of this symmetry. We have e.g. for $s = 3$ writing 1 and 0 for A and a:

$$\begin{aligned} p_1(1) &= p(111) + p(110) + p(101) + p(100) \\ &= p(111) + 2p(110) + p(100) = p(A^3) + 2p(A^2a) + p(Aa^2). \end{aligned}$$

Before deriving our recurrence formula for every s we shall determine here the *formula for* $s = 2$ by explicit computation, (a method which could not be generalized.) There are, for $s = 2$, the following six types of zygotes: $(A^2; A^2)$, $(A^2; Aa)$, $(A^2; a^2)$, $(Aa; Aa)$, $(Aa; a^2)$, $(a^2; a^2)$. We then derive by virtue of (2), (3), and (7):

$$\begin{aligned} p^{(n)}(A^2) &= w^{(n)}(A^2; A^2)\cdot 1 + 2w^{(n)}(A^2; Aa)(l_2 + l_1) \\ &\quad + 2w^{(n)}(Aa; A^2)(l_1 + l_0) + w^{(n)}(A^2; a^2) l_2 \end{aligned} \tag{12}$$

$$+ w^{(n)}(a^2; A^2)l_0 + 4w^{(n)}(Aa; Aa)l_1/2$$
$$= w^{(n)}(A^2; A^2) + 4w^{(n)}(A^2; Aa)(l_0 + l_1)$$
$$+ 2w^{(n)}(A^2; a^2)l_0 + 2w^{(n)}(Aa; Aa)l_1.$$

To understand this consider any arbitrary term of (12), e.g. $2w^{(n)}(A^2; Aa)(l_2 + l_1)$. The probability of a zygote to be of type $(A^2; Aa)$ is $2w^{(n)}(A^2; Aa)$ (since the A in the group after the ; can be the first or the second letter). Such a zygote transmits A^2, either by transmitting its two maternal values (with probability $\lambda_2 = l_2$) or by transmitting the paternal A-value and one of the two maternal A-values, the corresponding probability being $2\lambda_1 = l_1$. Next let us write the same formula for the $(n+1)^{st}$ generation, and observe that

$$w^{(n+1)}(x; y) = p^{(n)}(x)p^{(n)}(y). \tag{13}$$

We find, using (11):

$$p^{(n+1)}(A^2) = p^{(n)}(A^2)^2 + p^{(n)}(A^2)p^{(n)}(Aa)(4l_0 + 4l_1)$$
$$+ p^{(n)}(A^2)p^{(n)}(a^2)\cdot 2l_0 + p^{(n)}(Aa)^2\cdot 2l_1$$
$$= 2l_0[p^{(n)}(A^2)^2 + 2p^{(n)}(A^2)p^{(n)}(Aa) + p^{(n)}(A^2)p^{(n)}(a^2)]$$
$$+ 2l_1[p^{(n)}(A^2)^2 + 2p^{(n)}(A^2)p^{(n)}(Aa) + p^{(n)}(Aa)^2]$$
$$= 2l_0p^{(n)}(A^2) + 2l_1[p^{(n)}(A)]^2$$

Hence

$$p^{(n+1)}(A^2) = (l_0 + l_2)p^{(n)}(A^2) + 2l_1[p^{(n)}(A)]^2 \tag{14}$$

and in an analogous way we find:

$$p^{(n+1)}(Aa) = (l_0 + l_2)p^{(n)}(Aa) + 2l_1p^{(n)}(A)p^{(n)}(a) \tag{14'}$$

$$p^{(n+1)}(a^2) = (l_0 + l_2)p^{(n)}(a^2) + 2l_1[p^{(n)}(a)]^2. \tag{14''}$$

These are the desired recurrence formulae for $s = 2$.

Addition of (14) and (14') gives, since $l_0 = l_2$:

$$p^{(n+1)}(A^2) + p^{(n+1)}(Aa) = 2l_0[p^{(n)}(A^2) + p^{(n)}(Aa)]$$
$$+ 2l_1p^{(n)}(A)[p^{(n)}(A) + p^{(n)}(a)] = p^{(n)}(A).$$

Hence

$$p^{(n+1)}(A) = p^{(n)}(A) \qquad (n = 0, 1, 2, \cdots). \tag{15}$$

In the same way we conclude for $p^{(n)}(a)$ and thus find:

$$p^{(n)}(A) = p^{(0)}(A) \qquad (n = 1, 2, \cdots) \tag{15'}$$
$$p^{(n)}(a) = p^{(0)}(a).$$

The proportion of each allele is preserved through the generations. This, however, has been proved so far for $s = 2$ and two alleles only.

3. The general recurrence formula. It is obvious that for general s it is hardly possible to enumerate explicitely all $w^{(n)}$-terms entering in the expression for a certain $p^{(n)}$ the way it was done in (12). This is however not necessary because the concepts of the segregation distribution, of the distribution of gametes, and of the marginal distributions as explained in sections 1 and 2 afford an easy way to derive the recurrence relations for general s, practically without computations and in a way that generalizes easily to the case of multiple alleles.

Consider $p^{(n+1)}(A^z a^{s-z}) = p'(A^z a^{s-z})$. There are $\binom{s}{z}$ such probabilities and we want to express $\binom{s}{z} p'(A^z a^{s-z})$ in terms of the $w^{(n+1)} = w'$. Each w' will be multiplied by a sum of λ-values, since a gamete $(A^z a^{s-z})$ can be derived from *such* zygotes only which *possess* z A-values and $(s - z)$ a-values, (w-distribution), and also *transmit* them, (λ-distribution). We are going to compute the coefficient of λ_α $(\alpha = 0, 1, 2, \cdots s)$. According to its meaning such a λ_α is multiplied by a sum of such $w'(x; y)$ which contain before the semicolon α of the values $A^z a^{s-z}$ and after the semicolon $(s - \alpha)$ of these values. More briefly we may say each of the $w'(x; y)$ which multiply λ_α must contain α "correct" genes before and $(s - \alpha)$ correct ones after the semicolon, where we mean by "correct" values, values contained in $A^z a^{s-z}$. Hence such a w' will be of the form

$$w'(A^x a^{\alpha-x} \cdots ; \quad A^{z-x} a^{s-z-\alpha+x} \cdots)$$

where

$$0 \leqq x \leqq \alpha \text{ and } 0 \leqq z - x \leqq s - \alpha. \tag{16}$$

Now, after α places before and $(s - \alpha)$ places behind the semicolon have been occupied there remain still $(s - \alpha)$ available places before and α behind the semicolon, indicated by points; these places may be filled by any combination of A's and a's. Consider a fixed combination of this sort (e.g. the one where all available $(s - \alpha) + \alpha = s$ places are filled by A's.) How many such w' do exist? First, the α "correct" letters before the semicolon may be distributed over the s places before the semicolon in $\binom{s}{\alpha}$ ways and, likewise, the $(s - \alpha)$ "correct" values after the semicolon in $\binom{s}{s-\alpha} = \binom{s}{\alpha}$ ways. Moreover, these "correct" values can be arranged, each time, arbitrarily within their places; this can be done in $\binom{\alpha}{x}$ ways for the first and in $\binom{s-\alpha}{z-x}$ ways for the second set. The coefficient of any such w' is therefore $\binom{s}{\alpha}\binom{s}{\alpha}\binom{\alpha}{x}\binom{s-\alpha}{z-x}$. Using (13) we get, still concentrating on the coefficient of λ_α:

$$p'(A^z a^{s-z}) = \cdots + \lambda_\alpha \sum \left[\frac{\binom{s}{\alpha}^2 \binom{\alpha}{x}\binom{s-\alpha}{z-x}}{\binom{s}{z}} p(A^x a^{\alpha-x} \cdots) p(A^{z-x} a^{s-z-\alpha+x} \cdots) \right]$$

We may now assume, to fix the ideas, that the α occupied places in the first p to the right are e.g. the *first* α places and, likewise, the $(s-\alpha)$ places in the second p are the *first* $(s-\alpha)$ places.

Consider next the empty $(s-\alpha)+\alpha$ places, marked by points, in the preceding formula. On these places we have to write all possible combinations of the variables A and a and we have to take the sum of all these products. Now observe that in *each* of the two p's all possible combinations of the two letters have to be written on the empty places. It follows that *the double sum of all these products resolves into the product of the respective sums of p's.* In the first of these sums each term contains $A^x a^{\alpha-x}$ on the first α places and on the other places all possible combinations of the two letters. Hence, such a sum is precisely the *marginal distribution* $p_{12\ldots\alpha}(A^x a^{\alpha-x}) = p(A^x a^{\alpha-x})$, and an anologous statement holds for the second sum. Thus the coefficient of λ_α is simply

$$\frac{\binom{s}{\alpha}^2\binom{\alpha}{x}\binom{s-\alpha}{\alpha-x}\,p(A^x a^{\alpha-x})p(A^{s-x}a^{s-\alpha+x})}{\binom{s}{z}}.$$

Finally it is easily seen that

$$\binom{s}{\alpha}\binom{\alpha}{x}\binom{s-\alpha}{\alpha-x} = \binom{s}{z}\binom{z}{x}\binom{s-z}{\alpha-x}. \tag{17}$$

In fact in the binomial coefficients on the left side the lower number in each of the symbols is $\geqq 0$ and $\leqq$ the upper number because of $s \geqq \alpha \geqq 0$ and (16); and because of the equally obvious inequality $0 \leqq z \leqq s$ and (16) the symbols on the right of (17) are correct binomial coefficients too and the formal equality of both sides of (17) is seen directly:

$$\frac{s!}{\alpha!(s-\alpha)!}\,\frac{\alpha!}{x!(\alpha-x)!}\,\frac{(s-\alpha)!}{(z-x)!(s-\alpha-z+x)!} = \frac{s!\;z!\;(s-z)!}{z!(s-z)!x!(z-x)!(\alpha-x)!(s-z-\alpha+x)!}.$$

Hence the coefficient of λ_φ is simply $\binom{s}{\alpha}\binom{z}{x}\binom{s-z}{\alpha-x}$ and using $l_\alpha = \lambda_\alpha\binom{s}{\alpha}$ we get the elegant result:

$$p'(A^s a^{s-s}) = \sum_{\alpha=0}^{s} l_\alpha \sum_x \binom{z}{x}\binom{s-z}{\alpha-x}\,p(A^x a^{\alpha-x})p(A^{s-x}a^{s-\alpha+x}) \tag{18}$$

the range of x being given by (16).

This is the general recurrence formula which is of a remarkably clear structure. For *uniform* s.d. we have to put $l_\alpha = \binom{s}{\alpha}\Big/\binom{2s}{s}$.

For a *homozygous individual* where $s = z$ our formula becomes:

$$p'(A^s) = \sum_{\alpha=0}^{s} l_\alpha \binom{s}{\alpha}\,p(A^\alpha)p(A^{s-\alpha}) \tag{19}$$

since

$$\sum_z \binom{z}{x}\binom{s-z}{\alpha-x} = \binom{s}{\alpha}. \tag{20}$$

The biological probability-interpretation of (19) is simple: $l_\alpha \binom{s}{\alpha}$ is the probability that α maternal and $(s-\alpha)$ paternal genes be transmitted, $p(A^\alpha)$ is the probability of a maternal gamete with α A-chromosomes, and $p(A^{s-\alpha})$ is the analogous probability of a paternal gamete with $(s-\alpha)$ A-values in the n^{th} generation. The sum, over all α's, of the products of these three probabilities is obviously the probability of a gamete of type A^s in the following generation (if random breeding is assumed).

The coefficients $\binom{z}{x}\binom{s-z}{\alpha-x}$ of l_α in the more general formula (18) have an immediate meaning too: The x A-values in the first p-distribution to the right can be picked from the z A-values in the p'-distribution to the left in $\binom{z}{x}$ ways and, similarly, the $(s-z)$ a-values in p' in $\binom{s-z}{\alpha-x}$ ways. The interpretation then continues in the same way as for (19). It may be remarked that, while these recurrence laws are simple and basic, they are far from being trivial. We shall see that e.g. for chromatid segregation the recurrence law is of a quite different structure. To illustrate this we mention, without proof, the simplest of the recurrence formulae, the ones for $s = 2$, for "random chromatid segregation." It reads:

$$\begin{aligned} p^{(n+1)}(A^2) &= \tfrac{3}{4}p^{(n)}(A^2) + \tfrac{1}{4}p(A) + \tfrac{1}{4}p(A)^2 \\ p^{(n+1)}(Aa) &= \tfrac{3}{4}p^{(n)}(Aa) \qquad\qquad + \tfrac{1}{4}p(A)p(a). \end{aligned}$$

Returning to our formulae we write as examples the formulae for:

$$\begin{aligned} s = 3 \quad p'(A^3) &= 2l_0p(A^3) + 6l_1p(A^2)p(A) \\ p'(A^2a) &= 2l_0p(A^2a) + 2l_1[p(A^2)p(a) + 2p(Aa)p(A)] \end{aligned} \tag{21}$$

and for $s = 4$ (with different l_2 of course):

$$\begin{aligned} p'(A^4) &= 2l_0p(A^4) + 8l_1p(A)p(A^3) + 6l_2p(A^2)^2 \\ p'(A^3a) &= 2l_0p(A^3a) + 2l_1[p(A^3)p(a) + 3p(A^2a)p(A)] \\ &\qquad + 6l_2p(A^2)p(Aa) \\ p'(A^2a^2) &= 2l_0p(A^2a^2) + 4l_1[p(A)p(Aa^2) + p(a)p(A^2a)] \\ &\qquad + 2l_2[p(A^2)p(a^2) + 2p(Aa)^2]. \end{aligned} \tag{22}$$

Finally, for $s = 6$ we write the formula for $p'(A^3a^3)$:

$$p'(A^3a^3) = 2l_0p(A^3a^3) + 6l_1[p(A)p(A^2a^3) + p(a)p(a^2A^3)]$$

$$+ 6l_2[p(A^2)p(Aa^3) + 3p(Aa)p(A^2a^2) + p(a^2)p(A^3a)] \tag{23}$$
$$+ 2l_3[p(A^3)p(a^3) + 9p(A^2a)p(Aa^2)].$$

(We may add that the coefficients of the different p-products, their weights, can also be interpreted in the following way: Consider e.g. the l_2-group in (23). There are on the whole $\binom{6}{2} = 15$ terms in this group (in (23) $6l_2$ stands for $3l_2 + 3l_4$). These 15 terms are of the types $p(a^2)p(A^3a)$, $p(Aa)p(A^2a^2)$, and $p(A^2)p(Aa^3)$. In the first of the three products there are $\binom{2}{2}\binom{4}{1} = 1 \cdot 4 = 4$ possible arrangements of the arguments, in the second one $\binom{2}{1}\binom{4}{2} = 12$ and in the last one there are again four, and $4:12:4 = 1:3:1$, as is seen in (23). In the same way the $\binom{6}{3} = 20$ terms which multiply l_3 have to be distributed in the proportions $1 \cdot 1:3 \cdot 3 = 1:9$, as the formula confirms.)

By means of (18) the gametic proportions in any generation can be computed successively starting from some original distribution. The distribution of zygotes, i.e. of genotypes, follows then from (13). (It may be mentioned that Haldane has investigated the recurrence problem for $s = 2$ and $s = 3$ and that there is a mistake in his results for $s = 3$. Since his results may be used by biologists without being checked it seems adequate to indicate the difference: Haldane denotes by p_n, q_n, r_n, s_n the probabilities denoted by the author by $p^{(n)}(A^2)$, $p^{(n)}(A^2a)$, $\cdots$. With this notation and under Haldane's assumption of twenty equally likely cases his formula at the bottom of page 365, [5] should read:

$$q_{n+1} = q_n + \tfrac{3}{10}(4p_nq_n - 4q_n^2 + p_ns_n - q_nr_n + 2r_n^2 - 2q_ns_n)$$

instead of

$$q_{n+1} = q_n + \tfrac{3}{10}(4p_nq_n - 4q_ns_n + p_ns_n - q_nr_n + 2r_n^2 + 2q_ns_n).$$

There is no apparent regularity in this way to write the recurrence formula and thus the control consisted in direct computation and in the comparison with the second formula (21).),

The structure of (18) will become even clearer in case of several alleles.

4. Recurrence formulae for the marginal distributions. For the integration of the recurrence-system (18) as well as for the limit-theorem (section 5 and 6) we need *the* s.d. *corresponding to the marginal distributions, of order* 2, 3, $\cdots$ $(s-1)$, *which occur in* (18). We start with $s = 3$: Upon addition of the two formulae (21) we get

$$p'(A^2) = 2l_0p(A^3) + 6l_1p(A^2) + 4l_1p(A)^2$$
$$= (2l_0 + 2l_1)p(A^2) + 4l_1p(A)^2.$$

In the same way we add $p'(A^2a)$ and $p'(Aa^2)$ and get the two formulae

$$(24)\qquad \begin{aligned} s = 3\colon\ p'(A^2) &= l_0^{(1)}p(A^2) + 2l_1^{(1)}p(A)^2 + l_2^{(1)}p(A^2)^2 \\ p'(Aa) &= l_0^{(1)}p(Aa) + 2l_1^{(1)}p(A)p(a) + l_2^{(1)}p(Aa) \end{aligned}$$

where

$$(25)\qquad \begin{gathered} l_0^{(1)} = l_2^{(1)} = l_0 + l_1\,, \qquad l_1^{(1)} = l_1 + l_2 = 2l_1 \\ l_0^{(1)} + 2l_1^{(1)} + l_2^{(1)} = 1. \end{gathered}$$

For *arbitrary* s we get in the same way

$$(26)\qquad \begin{aligned} p'(A^{s-1}) &= l_0^{(1)}p(A^{s-1}) + (s-1)l_1^{(1)}p(A^{s-2})p(A) \\ &\quad + \binom{s-1}{2} l_2^{(1)}p(A^{s-3})p(A^2) + \cdots + l_{s-1}^{(1)}p(A^{s-1}) \end{aligned}$$

where

$$(27)\qquad l_0^{(1)} = l_0 + l_1\,, \qquad l_1^{(1)} = l_1 + l_2\,, \qquad l_2^{(1)} = l_2 + l_3\,, \cdots l_{s-1}^{(1)} = l_{s-1} + l_s\,.$$

In the same way we derive by a second summation a recurrence formula for $p'(A^{s-1}) + p'(A^{s-2}a) = p'(A^{s-2})$, the marginal distribution of order $(s-2)$:

$$(28)\qquad \begin{aligned} p'(A^{s-2}) &= l_0^{(2)}p(A^{s-1}) + (s-2)l_1^{(2)}p(A^{s-3})p(A) \\ &\quad + \binom{s-2}{2} l_2^{(2)}p(A^{s-4})p(A^2) + \cdots + l_{s-2}^{(2)}p(A^{s-2}) \end{aligned}$$

where

$$(29)\qquad l_0^{(2)} = l_0 + 2l_1 + l_2\,, \qquad l_1^{(2)} = l_1 + 2l_2 + l_3\,, \qquad l_2^{(2)} = l_2 + 2l_3 + l_4\,, \cdots$$

and, in general, *for the marginal distribution of order* $(s-\nu)$, found by ν-tuple summation:

$$(30)\qquad \begin{aligned} p'(A^{s-\nu}) &= l_0^{(\nu)}p(A^{s-\nu}) + \binom{s-\nu}{1} l_1^{(\nu)}p(A^{s-\nu-1})p(A) \\ &\quad + \binom{s-\nu}{2} l_2^{(\nu)}p(A^{s-\nu-2})p(A^2) + \cdots \binom{s-\nu}{s-\nu} l_{s-\nu}^{(\nu)}p(A^{s-\nu}) \end{aligned}$$

where

$$(31)\qquad \begin{aligned} l_0^{(\nu)} &= l_0 + \nu l_1 + \binom{\nu}{2} l_2 + \cdots + \binom{\nu}{\nu} l_\nu\,, \\ l_1^{(\nu)} &= l_1 + \binom{\nu}{1} l_2 + \binom{\nu}{2} l_3 + \cdots l_{\nu+1}\,, \\ &\cdots. \end{aligned}$$

Hence we have

$$(32)\qquad l_a^{(\nu)} = \binom{\nu}{0} l_a + \binom{\nu}{1} l_{a+1} + \binom{\nu}{2} l_{a+2} + \cdots + \binom{\nu}{\nu} l_{a+\nu}$$

which may be written symbolically

$$l_\alpha^{(\nu)} = (1 + l_\alpha)^\nu \text{ with } l_\alpha^i = l_{\alpha+i} \qquad (i = 0, 1, \cdots \nu). \tag{32'}$$

Obviously

$$l_\alpha^{(\nu)} = l_{s-\nu-\alpha}^{(\nu)} \text{ because of } l_\alpha = l_{s-\alpha}, \tag{33}$$

and

$$\sum_{\alpha=0}^{s-\nu} \binom{s-\nu}{\alpha} l_\alpha^{(\nu)} = 1 \tag{34}$$

follows from (7).

For uniform distribution $\binom{2s}{s} l_\alpha = \binom{s}{\alpha}$, (32) becomes

$$\binom{2s}{s} l_\alpha^{(\nu)} = \binom{\nu}{0}\binom{s}{\alpha} + \binom{\nu}{1}\binom{s}{\alpha+1} + \cdots + \binom{\nu}{\nu}\binom{s}{\alpha+\nu} = \binom{s+\nu}{\alpha+\nu}. \tag{35}$$

In fact

$$\sum_{i=0}^{\nu} \binom{\nu}{i}\binom{s}{\alpha+i} = \binom{s+\nu}{\alpha+\nu}, \tag{36}$$

as can be seen from the expansion of the identity

$$(a + c)^\nu (a + c)^s = (a + c)^{\nu+s}$$

for $\nu = 0$ we get

$$l_\alpha^{(0)} = l_\alpha. \tag{32''}$$

Hence we have in (30) a recurrence formula for $p'(A^{s-\nu})$ which for $\nu = 0$ reduces to (19) by virtue of (32''). The analogous recurrence formula for non homozygous individuals will be derived in section 8 when multiple alleles are considered. The result, for two alleles is the following:

$$p'(A^{s-\nu}a^{s-s}) = \sum_{\alpha=0}^{s-\nu} l_\alpha^{(\nu)} \sum_x \binom{z-\nu}{x}\binom{s-z}{\alpha-x} p(A^z a^{\alpha-x}) p(A^{s-\nu-x} a^{s-s-\alpha+x})$$

where the range of x is such that the symbols $\binom{z-\nu}{x}, \binom{s-z}{\alpha-x}$ are regular binomial coefficients.

5. The limit theorems. We shall use the following easily proved lemma which belongs to general analysis:

A) *Let* $x_{n+1} - \alpha x_n = y_n$ *where* $|\alpha| < 1$. *If* $\lim_{n\to\infty} y_n = y$, *then*

$$\lim_{n\to\infty} x_n = \frac{y}{1-\alpha}.$$

Secondly we need the following lemma:

B) *With the definitions* (32) *the equality*

$$l_0^{(\nu)} = l_{s-\nu}^{(\nu)} = \tfrac{1}{2} \qquad (\nu = 0, 1, \cdots (s-2))$$

holds, if and only if $l_0^{(0)} \equiv l_0 = l_s = \tfrac{1}{2}$.

In fact we get from (32)

$$(32''') \qquad l_0^{(\nu)} = l_0 + \nu l_1 + \binom{\nu}{2} l_2 + \cdots + \binom{\nu}{\nu} l_\nu$$

besides we have by definition of the l_α

$$(37) \qquad 1 = l_0 + s l_1 + \binom{s}{2} l_2 + \cdots + \binom{s}{s} l_s .$$

Hence, firstly, if $l_0 = l_s = \tfrac{1}{2}$ we see from (7) that $l_1 = l_2 = \cdots = l_{s-1} = 0$, hence $l_0^{(\nu)} = l_0 = \tfrac{1}{2}$. On the other hand we see from (32''') that

$$l_0^{(0)} < l_0^{(1)} < l_0^{(2)} \cdots < l_0^{(s-2)} < l_0^{(s-1)}$$

unless all l_i, except l_0 are zero. Now we shall prove presently that $l_0^{(s-1)} = \tfrac{1}{2}$. If this is assumed for the moment it follows that $l_0^{(\nu)} < \tfrac{1}{2}$ $(\nu = 0, 1, \cdots (s-2))$ unless all l_i except $l_0 = l_s$ vanish, i.e. unless $l_0 = \tfrac{1}{2}$. Thus lemma B) will be proved if we can show that

$$(38) \qquad l_0^{(s-1)} = \tfrac{1}{2} .$$

In fact, because of $l_i = l_{s-i}$:

$$\begin{aligned} l_0^{(s-1)} &= l_0 + (s-1) l_1 + \binom{s-1}{2} l_2 + \binom{s-1}{3} l_3 \\ &\qquad + \cdots + \binom{s-1}{3} l_{s-4} + \binom{s-1}{2} l_{s-3} + \binom{s-1}{1} l_{s-2} + l_{s-1} \\ &= l_0 + s l_1 + \binom{s}{2} l_2 + \cdots + \binom{s}{\mu} l_\mu \cdot \tfrac{1}{2} \quad \text{if} \quad s = 2\mu \\ &= l_0 + s l_1 + \binom{s}{2} l_2 + \cdots + \binom{s}{\mu} l_\mu \quad \text{if} \quad s = 2\mu + 1 \end{aligned}$$

but, because of (7) either expression equals $\tfrac{1}{2}$. Thus (38) is proved and B) holds true.

From (38) we now first derive the following limit theorem, (for the diploid this is G. H. Hardy's result quoted in the introduction).

For every n and s we have

$$(39) \qquad p^{(n)}(A) = p^{(0)}(A), \qquad p^{(n)}(a) = p^{(0)}(a) \qquad (n = 1, 2, \cdots)$$

where $p^{(n)}(A)$ *and* $p^{(n)}(a)$ *are the marginal distributions of order one of* $p^{(n)}(A^z a^{s-z})$ $(z = 0, 1, \cdots s)$. *Hence in a* $2s$*-ploid, just as in a diploid, the gametic proportions of the dominant and the recessive gene remain constant through the generations.*

In fact for $\nu = s - 1$ we get from (33) and (38) $l_0^{(s-1)} \equiv l_1^{(s-1)} = \frac{1}{2}$ and (39) follows from (30).

Now let us investigate *the limit behaviour of* $p^{(n)}(A^x a^{s-x})$. Assume $2l_0 < 1$ and consider for the given s the marginal distributions of order two: $p^{(n)}(A^2), p^{(n)}(Aa)$, and $p^{(n)}(a^2)$ for which we have:

$$p^{(n+1)}(A^2) = 2l_0^{(s-2)} p^{(n)}(A^2) + 2l_1^{(s-2)} p(A)^2.$$

We know that $l_0^{(s-2)} < \frac{1}{2}$ because of $l_0 < \frac{1}{2}$ and lemma B). Then, by lemma A), since $2l_1^{(s-2)} p(A)^2$ is independent of n we get

$$\lim_{n \to \infty} p^{(n)}(A^2) = \frac{2l_1^{(s-2)} p(A)^2}{1 - 2l_0^{(s-2)}} = p(A)^2$$

and the analogous relations hold for $p^{(n)}(Aa)$ and $p^{(n)}(a^2)$. Next, we have for the marginal distributions of order three, e.g. for $p^{(n)}(A^3)$:

$$p^{(n+1)}(A^3) = 2l_0^{(s-3)} p^{(n)}(A^3) + 6l_1^{(s-3)} p^{(n)}(A^2) p(A)$$

which is again of the form $x_{n+1} = \alpha x_n + y_n$. Now, because of the immediately preceding result the limit of y_n is $6l_1^{(s-3)} p(A)^3$ and since by lemma B) $2l_0^{(s-3)} < 1$, we get from A)

$$\lim_{n \to \infty} p^{(n)}(A^3) = p(A)^3$$

and similar results for $p^{(n)}(A^2a)$, $p^{(n)}(Aa^2)$, and $p^{(n)}(a^3)$. Continuing in this way we see that

$$\lim_{n \to \infty} p^{(n)}(A^x a^{s-x}) = p(A)^x p(a)^{s-x} \tag{41}$$

under the only hypothesis

$$l_0 + l_s < 1. \tag{42}$$

If, however $l_0 + l_s = 1$, we have

$$p^{(n)}(A^x a^{s-x}) = p^{(0)}(A^x a^{s-x}) \qquad (n = 1, 2, \cdots). \tag{43}$$

Thence we get the theorem:

Denote by $l_0 + l_s$ *the probability that in the propagation of a* $2s$*-ploid every gamete consist entirely of paternally derived genes or entirly of maternally derived ones. Unless this probability equals one, i.e. unless all kinds of "mixed" gametes are a priori excluded, the distribution of gametes tends towards the limit* (41) *where the genes are independently distributed. If, however* $l_0 + l_s = 1$ *the distribution of gametes remains constant through the generations according to* (43).

The corresponding result for the distribution of zygotes (of genotypes) is the following:

If (42) *holds the distribution of genotypes* $w^{(n)}(A^x a^{s-x}; A^y a^{s-y})$ *converges towards* $p(A)^{x+y} p(a)^{2s-x-y}$. *If, however,* $l_0 + l_s = 1$ *the distribution of zygotes does not change after the first filial generation,* i.e.

$$w^{(n)} = w^{(1)} \qquad (n = 1, 2, \cdots). \tag{43'}$$

Haldane has proved that, with random chromosome segregation the distribution to the right of (41) is stable, but not that this stable distribution is ever reached.

Theorems analogous to the above but for multiple alleles will be derived in sec. 8.

6. Integration of the recurrence formulae. We wish to solve the recurrence formulae (18) so as to find $p^{(n)}(A^x a^{s-x})$ explicitely in terms of "n", of the initial values $p^{(0)}(z)$, and, of course, of the parameters l_α. It will be sufficient to consider the system (19), i.e. the homozygous case.—These formulae present themselves as a system of quadratic difference equations with constant coefficients. In fact, each marginal distribution at the right in (19) is, by definition, a sum of p-values (e.g. for $s = 4$: $p^{(n)}(A^3) = p^{(n)}(A^4) + p^{(n)}(A^3a)$, or $p^{(n)}(A^2) = p^{(n)}(A^4) + 2p^{(n)}(A^3a) + p^{(n)}(A^2a^2)$) and the right side of (19) consists of a sum of products of two such marginal distributions. Upon reintroduction of the original $p^{(n)}$ the simple regularity of the recurrence formulae vanishes (see e.g. Haldane's formula for $s = 3$, p. 365, [5]) it seems that in this way an integration procedure is hard to work out.

On the other hand the situation looks different if we use throughout the marginal distribution as such, and our *recurrence relations for these marginal distributions*, as derived in sec. 4. It is particularly helpful that these recurrence formulae are of exactly the same structure as the original recurrence formulae, the only difference being that they contain other constant coefficients; this change of constants is a systematic one: It merely consists in the fact that in a recurrence formula which corresponds to a ν-tupple summation $l_\alpha^{(\nu)}$ stands in place of l_α; this $l_\alpha^{(\nu)}$ is given by (32) in terms of the original l_α.

If these remarks are used the problem stands as follows: As in the preceding section we may write (19) in the form

$$x_{n+1} = ax_n + f_n$$

$$x_n = p^{(n)}(A^s), \qquad a = 2l_0, \qquad f_n = \sum_{\alpha=0}^{s-1} l_\alpha \binom{s}{\alpha} p^{(n)}(A^\alpha) p^{(n)}(A^{s-\alpha}). \tag{44}$$

In f_n there occur marginal distributions of order $s_1 < s$ only, i.e. of orders $1, 2, \cdots, (s-1)$. If we know f_n the solution of (44) is:

$$x_n = a^n x_0 + \sum_{\nu=1}^{n-1} f_\nu a^{n-1-\nu}. \tag{45}$$

For each of the marginal distributions in f_n, of order s_1, again a difference equation (44) holds, (with a different "a" and a different f_n) which can be solved if the corresponding f_n is known. This f_n depends on marginal distributions of order $s_2 < s_1$ and if we can determine them explicitly if we know this new f_n, and so it goes on, down to marginal distributions of order two where the recurrence formulae become trivial since the right side does not depend on n any more.

It ought to be stressed that each result, i.e. the explicit expression of $p^{(n)}(A^s)$ for some "s", can be used again and again. As an illustration assume that we

know explicit formulae for $s = 2, 3$, and 4 and that we want the solution for $s = 6$. We assume that each of the formulae for $s = 2, 3, 4$ contains merely those l_α which correspond to this particular "s", i.e., the formula for $s = 4$ is written in terms of those l_α for which $\sum_{\alpha=0}^{4} \binom{s}{\alpha} l_\alpha = 1$. In order to find a solution for $s = 5$ we write (19)

$$\text{(a)} \quad p^{(n+1)}(A^5) - 2l_0 p^{(n)}(A^5) = 10l_1 p^{(n)}(A^4)p(A) + 20l_2 p^{(n)}(A^3)p^{(n)}(A^2).$$

Here $2l_0 + 10l_1 + 20l_2 = 1$. In (a) the expressions $p^{(n)}(A^s)$ for $s = 2, 3$, and 4 are known by hypothesis; we have only to write the expression for $p^{(n)}(A^4)$ with constants $l_\alpha^{(1)}$, that for $p^{(n)}(A^3)$ with $l_\alpha^{(2)}$, and the last one with $l_\alpha^{(3)}$. Thus the right side of (a) is a known function of "n" and, using (45) we determine $p^{(n)}(A^5)$ in terms of "n", of l_α, $(\alpha = 0, \cdots 5)$, of $l_\alpha^{(1)}$ $(\alpha = 0, \cdots 4)$, of $l_\alpha^{(2)}$, $(\alpha = 0, \cdots 3)$, and of $l_\alpha^{(3)}$ $(\alpha = 0, \cdots 2)$. Finally we replace the $l_\alpha^{(1)}$, $l_\alpha^{(2)}$, $l_\alpha^{(3)}$ by the l_α $(\alpha = 0, \cdots 5)$, using (32), and possess thus, once and for ever, a formula for $s = 5$. This we may use, in turn, to determine $p^{(n)}(A^6)$ proceeding in the same way as before.

We can go a step further in the systematization of the procedure. First from what we said so far we know obviously the *general form* of our solution. It will be of the form:

$$\begin{aligned} p^{(n)}(A^s) = {} & \alpha_s^{(n)} p(A^s) + [\alpha_{s-1,1}^{(n)} p(A^{s-1})p(A) + \alpha_{s-2,2}^{(n)} p(A^{s-2})p(A^2) + \cdots] \\ & + [\alpha_{1,1,s-2}^{(n)} p(A^{s-2})p(A)^2 + \cdots] + [\alpha_{1,1,1,s-3}^{(n)} p(A^{s-3})p(A)^3 + \cdots] \\ & + \cdots + \alpha_{1,1,\cdots 1}^{(n)} p(A)^s. \end{aligned} \tag{46}$$

Here p stands for $p^{(0)}$ and the coefficients $\alpha_{i,j,\cdots k}^{(n)}$ depend on n and on the l_α. In (46) the first bracket contains all possible products of *two* factors $p(A^x)p(A^{s-x})$, the second, products of *three* factors $p(A^x)p(A^y)p(A^{s-x-y})$ where x, y, and $s - x - y$ are integers $\geqq 1$, etc. We have e.g. for $s = 6$:

$$\begin{aligned} p^{(n)}(A^6) = {} & \alpha_6^{(n)} p(A^6) + [\alpha_{51}^{(n)} p(A^5)p(A) + \alpha_{42}^{(n)} p(A^4)p(A^2) + \alpha_{33}^{(n)} p(A^3)^2] \\ & + [\alpha_{114}^{(n)} p(A^4)p(A)^2 + \alpha_{123}^{(n)} p(A^3)p(A^2)p(A) + \alpha_{222}^{(n)} p(A^2)^3] \\ & + [\alpha_{1113}^{(n)} p(A^3)p(A)^3 + \alpha_{1122}^{(n)} p(A^2)^2 p(A)^2] \\ & + \alpha_{11112}^{(n)} p(A^2)p(A)^4 + \alpha_{111111}^{(n)} p(A)^6. \end{aligned} \tag{46}$$

All we need are the coefficients $\alpha_{i,j,\cdots k}^{(n)}$. Let us explain the procedure by means of the case $s = 6$. As an abbreviation put

$$2l_\alpha \binom{s}{\alpha} = v_\alpha, \qquad 2\binom{s-\nu}{\alpha} l_\alpha^{(\nu)} = v_\alpha^{(\nu)}. \tag{47}$$

That means e.g. for $s = 5$, $2l_0 + 10l_1 + 20l_2 = 1$ or $v_0 + v_1 + v_2 = 1$;

$s = 5, \nu = 1$: $2l_0^{(1)} + 8l_1^{(1)} + 6l_2^{(1)} = 1$ or $v_0^{(1)} + v_1^{(1)} + \frac{1}{2}v_2^{(1)} = 1$;

$s = 5, \nu = 2$: $2l_0^{(2)} + 6l_1^{(2)} = 1$, or $v_0^{(2)} + v_1^{(2)} = 1$;

$s = 5, \nu = 3$: $2l_0^{(3)} + 2l_1^{(3)} = 1$ or $v_0^{(3)} + \frac{1}{2}v_1^{(3)} = 1$.

With this notation (19) reads for $s = 6$:

$$p^{(n+1)}(A^6) = v_0 p^{(n)}(A^6) + v_1 p^{(n)}(A^5)p(A) + v_2 p^{(n)}(A^4)p^{(n)}(A^2) + \tfrac{1}{2}v_3[p^{(n)}(A^3)]^2. \tag{48}$$

We also need the formulae (46) for the marginal distributions of order 5, 4, 3, 2: They are:

$$p^{(n)}(A^5) = \alpha_5^{(n)}p(A^5) + [\alpha_{41}^{(n)}p(A^4)p(A) + \alpha_{32}^{(n)}p(A^3)p(A^2)] + [\alpha_{113}^{(n)}p(A^3)p(A)^2 + \alpha_{122}^{(n)}p(A^2)^2p(A)] + \alpha_{1112}^{(n)}p(A^2)p(A)^3 + \alpha_{11111}^{(n)}p(A)^5 \tag{46'}$$

$$p^{(n)}(A^4) = \alpha_4^{(n)}p(A^4) + \alpha_{13}^{(n)}p(A^3)p(A) + \alpha_{22}^{(n)}p(A^2)^2 + \alpha_{112}^{(n)}p(A^2)p(A)^2 + \alpha_{1111}^{(n)}p(A)^4 \tag{46''}$$

$$p^{(n)}(A^3) = \alpha_3^{(n)}p(A^3) + \alpha_{12}^{(n)}p(A^2)p(A) + \alpha_{111}^{(n)}p(A)^3 \tag{46'''}$$

$$p^{(n)}(A^2) = \alpha_2^{(n)}p(A^2) + \alpha_{11}^{(n)}p(A)^2. \tag{46^{iv}}$$

In each of these formulae the sum of all α is one, e.g.

$$\alpha_4^{(n)} + \alpha_{13}^{(n)} + \alpha_{22}^{(n)} + \alpha_{112}^{(n)} + \alpha_{1111}^{(n)} = 1,$$

etc. The recurrence relations (30) for these marginal distributions are:

$$p^{(n+1)}(A^5) = v_0^{(1)}p^{(n)}(A^5) + v_1^{(1)}p^{(n)}(A^4)p(A) + v_2^{(1)}p^{(n)}(A^3)p^{(n)}(A^2) \tag{48'}$$

$$p^{(n+1)}(A^4) = v_0^{(2)}p^{(n)}(A^4) + v_1^{(2)}p^{(n)}(A^3)p(A) + \tfrac{1}{2}v_2^{(2)}p^{(n)}(A^2)^2 \tag{48''}$$

$$p^{(n+1)}(A^3) = v_0^{(3)}p^{(n)}(A^3) + v_1^{(3)}p^{(n)}(A^2)p(A) \tag{48'''}$$

$$p^{(n+1)}(A^2) = v_0^{(4)}p^{(n)}(A^2) + \tfrac{1}{2}v_1^{(4)}p(A)^2. \tag{48^{iv}}$$

If we now substitute in (48) for $p^{(n)}(A^6)$ and for $p^{(n+1)}(A^6)$ the expression (46) and for $p^{(n)}(A^5), \cdots p^{(n)}(A^2)$ the expressions (46′) to (46^{iv}) we find recurrence formulae for the coefficients $\alpha_6^{(n)}, \cdots \alpha_{111111}^{(n)}$; $\alpha_5^{(n)}, \cdots \alpha_{11111}^{(n)}, \cdots \alpha_2^{(n)}, \alpha_{11}^{(n)}$, by equating coefficients of $p(A^6)$, of $p(A^5)p(A)$, etc., etc.

These recurrence relations for the $\alpha_{i,j,\cdots k}^{(n)}$ which are of a very clear structure are all we need. Let us consider them. First we can indicate immediately the value of $\alpha_i^{(n)}(i = 2, 3, \cdots 6)$. In fact we have

$$\alpha_6^{(n+1)} = v_0\alpha_6^{(n)}, \qquad \alpha_5^{(n+1)} = v_0^{(1)}\alpha_5^{(n)}, \cdots.$$

Hence

$$\alpha_6^{(n)} = (v_0)^n, \qquad \alpha_5^{(n)} = (v_0^{(1)})^n, \cdots \alpha_2^{(n)} = (v_0^{(4)})^n. \tag{49}$$

The other relations are of the form:

$$\alpha_{ij\cdots k}^{(n+1)} = v_0\alpha_{ij\cdots k}^{(n)} + A_{ij\cdots k}^{(n)} \tag{50}$$

where $A_{ij\cdots k}^{(n)}$ is composed in the following way: It is a sum of products of two α-values and one v-value: The α-values are such that their subscripts considered

together contain each of the figures $i, j, \cdots k$ which appear in $\alpha_{ij\cdots k}^{(n)}$ just once. The subscript of the v-value is equal to the sum of the subscripts of either α-value. (It makes no difference which of these sums is taken because of the symmetry property of the v's.) In this rule *it has to be understood that* $\alpha_1^{(n)} = 1$. We have e.g., if $s = 8$:

$$A_{11222}^{(n)} = v_1\alpha_1^{(n)}\alpha_{1222}^{(n)} + v_2(\alpha_2^{(n)}\alpha_{122}^{(n)} + \alpha_{11}^{(n)}\alpha_{222}^{(n)}) + v_3\alpha_{12}^{(n)}\alpha_{122}^{(n)} + v_4\alpha_{22}^{(n)}\alpha_{112}^{(n)}$$

(here we have $v_1 = v_7$, $v_2 = v_6$, $v_3 = v_5$). It is to be noted that if in $A_{ij\cdots k}^{(n)}$ a *square* of α's appears this term is multiplied by $\frac{1}{2}$. (See (48), (48″), (48$^{\text{iv}}$).)

Equations (50) are of form (44) and because of the obvious condition of $x_0 = 0$ they have the solution

$$x_n = \sum_{\nu=0}^{n-1} f_\nu a^{n-1-\nu}. \tag{45′}$$

To determine the $A_{ij\cdots k}^{(n)}$ on the right side of (50), we start with $\alpha_2^{(n)}$ and $\alpha_{11}^{(n)} = 1 - \alpha_2^{(n)}$. Next we determine $\alpha_3^{(n)}$, $\alpha_{12}^{(n)}$, and $\alpha_{111}^{(n)} = 1 - \alpha_3^{(n)} - \alpha_{12}^{(n)}$ making use of our result regarding $\alpha_2^{(n)}$, and so on. (As stated before: each formula must be used in terms of the corresponding v_α; e.g. if $s = 3$ we understand by v_α the values defined by $l_0 + l_3 = 2l_0 = v_0$, $3l_1 + 3l_2 = 6l_1 = v_1$, $v_0 + v_1 = 1$.)

As an illustration let us consider the $\alpha^{(n)}$ for $s = 2$, $s = 3$, $s = 4$: We first have

$$\alpha_2^{(n)} = (2l_0)^n, \qquad \alpha_{11}^{(n)} = 1 - (2l_0)^n. \tag{51}$$

Hence

$$p^{(n)}(A^2) = (2l_0)^n p(A^2) + [1 - (2l_0)^n]p(A)^2 \tag{52}$$

which may be written

$$p^{(n)}(A^2) = p(A)^2 + (2l_0)^n[p(A^2) - p(A)^2] \tag{52′}$$

or introducing

$$p^{(n)}(x_1x_2) - p(x_1)p(x_2) = q^{(n)}(x_1x_2) \qquad (x_i = A, a) \tag{52″}$$

simply

$$q^{(n)}(x_1x_2) = (2l_0)^n q(x_1x_2), \qquad \begin{pmatrix} x_1 = A, a \\ x_2 = A, a \end{pmatrix}. \tag{52‴}$$

Next consider $s = 3$ with the recurrence formula:

$$\alpha_{21}^{(n+1)} = v_0\alpha_{21}^{(n)} + \alpha_2^{(n)}\alpha_1^{(n)}v_1 \tag{53}$$

which gives, since $\alpha_1^{(n)} = 1$:

$$\alpha_{21}^{(n)} = v_1\sum_{\nu=0}^{n-1} v_0^{n-1-\nu}\alpha_2^{(\nu)} = v_1\sum_{\nu=0}^{n-1} v_0^{n-1-\nu}(v_0^{(1)})^\nu$$

$$= v_1\frac{(v_0^{(1)n} - v_0^n}{v_0^{(1)} - v_0} = \frac{6l_1}{(2l_0 + 2l_1) - 2l_0}[(2l_0 + 2l_1)^n - 2l_0)^n].$$

Hence, for $s = 3$:

$$\alpha_3^{(n)} = (2l_0)^n$$

$$\alpha_{21}^{(n)} = 3[(2l_0 + 2l_1)^n - (2l_0)^n] \tag{53'}$$

$$\alpha_{111}^{(n)} = 1 - \alpha_3^{(n)} - \alpha_{21}^{(n)} .$$

Hence the simple result:

$$p^{(n)}(A^3) = p(A^3) + (2l_0)^n[p(A^3) - p(A)^3] + 3[(2l_0 + 2l_1)^n - (2l_0)^n]p(A)[p(A^2) - p(A)^2] \tag{54}$$

and:

$$p^{(n)}(A^2a) = p(A^2)p(a) + (2l_0)^n[p(A^2a) - p(A)^2p(a)] + [(2l_0 + 2l_1)^n - (2l_0)^n]\{p(a)[p(A^2) - p(A)^2] + 2[p(Aa) - p(A)p(a)]\} \tag{54'}$$

etc.

For random chromatid segregation the rate of approach to equilibrium is in this case $(2/5)^n$ as found by Haldane. Our formula becomes in this case

$$p^{(n)}(A^3) = p(A)^3 + (\tfrac{1}{10})^n[p(A^3) - p(A)^3] + 3[(\tfrac{2}{5})^n - (\tfrac{1}{10})^n]p(A)[p(A^2) - p(A)^2] \tag{54''}$$

The formulae which take the place of (54′) in Haldane's paper are complicated and, as far as I can see, not correct.

For $s = 4$ we need the coefficients $\alpha_4^{(n)}$, $\alpha_{13}^{(n)}$, $\alpha_{22}^{(n)}$, $\alpha_{211}^{(n)}$. We find, with the v_a and l_a, that correspond to $s = 4$:

$$\alpha_{13}^{(n+1)} = v_0\alpha_{13}^{(n)} + v_1\alpha_1^{(n)}\alpha_3^{(n)}$$

$$\alpha_{13}^{(n)} = v_1 \sum_{r=0}^{n-1} v_0^{n-1-r}\alpha_3^{(r)} = v_1 \sum_{r=0}^{n-1} v_0^{n-1-r}(2l_0^{(1)})^r$$

$$= 8l_1 \frac{(2l_0 + 2l_1)^n - (2l_0)^n}{2l_1} = 4[(2l_0 + 2l_1)^n - (2l_0)^n]$$

From this derivation we conclude that we have for *any* s:

$$\alpha_{1,s-1}^{(n)} = s[(2l_0 + 2l_1)^n - (2l_0)^n] \tag{55}$$

so that $\alpha_s^{(n)}$ and $\alpha_{1,s-1}^{(n)}$ are known for every s-value.

These examples should suffice to illustrate our procedure. The computations and results become lengthy as s increases, but in principle there is no difficulty. There are a few more regularities which are helpful for the carrying through of the computations. It seems however not necessary to go into more details unless there is a demand for explicit expressions for higher s-values. At any rate, thanks to our recurrence formulae for the marginal distributions we possess a straight-forward and systematic *chain-procedure* instead of the original complicated quadratic system of difference equations.—It will depend on the actual problem to decide whether it is preferable to work out for a given s this explicit

solution which then holds for any n, or to compute the $p^{(n)}$ for a few successive generations by means of (19).

In case of non homozygous gametes the problem remains the same. In fact the same $\alpha_{ij\cdots k}^{(n)}$ appear. But each $\alpha_{ij\cdots k}^{(n)}$ is multiplied not by *one* product of p-values but by several such products with easily indicated weights (see e.g. (54′)). We shall consider this in the last section where we deal with multiple alleles.

Let us now determine the *rate of approoch to the equilibrium status* which we considered in the preceding section. Consider (45) and (45′) and observe that the coefficient "a" in these formulae is given in turn by v_0, $v_0^{(1)}$, $\cdots$ $v_0^{(s-2)}$ where each of these is greater than the preceding one (see (32‴)), if the degenerate case $l_0 = \frac{1}{2}$ is excluded. We see that, in order to know the rate of approach we have to know $v_0^{(s-2)}$, $(s \geqq 2)$, since this rate is of order $(v_0^{(s-2)})^n$. For uniform s.d. we get simply, from (32‴):

$$v_0^{(s-2)} = 2\frac{\binom{2s-2}{s}}{\binom{2s}{s}} = \frac{2s(s-1)}{2s(2s-1)} = \frac{s-1}{2s-1} \tag{56}$$

Thence: *The rate of approach to equilibrium is of order* $\left(\frac{s-1}{2s-1}\right)^n$ *in case of a uniform s.d. In general the rate is* $(2l_0^{(s-2)})^n$ *which may be faster or slower than* $\left(\frac{s-1}{2s-1}\right)^n$.

We can easily make a more detailed statement. Always using $s = 6$ as a paradigma the setup (46) may be written in the form:

$$\begin{aligned} p^{(n)}(A^6) = p(A)^6 &+ \alpha_6^{(n)}[p(A^6) - p(A)^6] + \alpha_{51}^{(n)}p(A)[p(A^5) - p(A)^5] \\ &+ \alpha_{42}^{(n)}[p(A^4)p(A^2) - p(A)^6] + \alpha_{33}^{(n)}[p(A^3)^2 - p(A)^6] \\ &+ \cdots + \alpha_{11112}^{(n)}p(A)^4[p(A^2) - p(A)^2] \end{aligned}$$

In this formula each of the $\alpha_{ij\cdots k}^{(n)}$ tends towards zero as $n \to \infty$, but at very different rates. We can indicate these rates explicitly. The slowest is that of $\alpha_{11112}^{(n)}$ since it depends in turn on $\alpha_{1112}^{(n)}$, $\cdots$ $\alpha_2^{(n)}$ and the rate of approach to zero of this last one is $(5/11)^n$ as just proved. On the other hand the term $p(A^6) - p(A)^6$ which somehow differs most from the equilibrium status is the one to vanish fastest, since $\alpha_6^{(n)} = (v_0)^n$, where $v_0 = 2 \Big/ \binom{2s}{s}$. Briefly we have, always, for $s = 6$ and for uniform s.d.:

Corresponding to the α-coefficient of:

$p(A^2) - p(A)^2$ a rate of approach to zero of: $\left(\frac{s-1}{2s-1}\right)^n = \left(\frac{5}{11}\right)^n$

$p(A^3) - p(A)^3$ ..... $\left[\frac{(s-1)(s-2)}{(2s-1)(2s-2)}\right]^n = \left(\frac{2}{11}\right)^n$

$$p(A^4) - p(A)^4 \ldots\ldots \left[\frac{(s-1)(s-2)(s-3)}{(2s-1)(2s-2)(2s-3)}\right]^n = \left(\frac{2}{33}\right)^n$$

$$\ldots\ldots\ldots\ldots$$

$$p(A^6) - p(A)^6 \ldots\ldots \left[2 \Big/ \binom{2s}{s}\right]^n = \left(\frac{1}{462}\right)^n$$

also corresponding to

$$p(A^3)^2 - p(A)^6 \ldots\ldots (v_0^{(3)})^{2n} = \left(\frac{2}{11}\right)^{2n} = \left(\frac{4}{121}\right)^n$$

$$p(A^4)p(A^2) - p(A)^6 \ldots\ldots (v_0^{(3)n})(v_0^{(4)})^n = \left(\frac{5}{11}\cdot\frac{2}{33}\right)^n$$

Thus, if we make a precise assumption regarding the s.d. we can predict, and therefore check, the rate of approach to equilibrium.

7. **Multiple Alleles. Definitions.** So far we studied the most frequently considered and perhaps most important case of $r = 2$ alleles. We shall now deal with *a 2s-ploid with r alleles*, the number, m, of factors being still equal to one.

Two ways of notation present themselves. We either take as starting point the r alleles, $a_1, \cdots a_r$ and indicate,—in the zygote as well as in the gamete,—the power of each a_i. That is the way we worked so far for $r = 2$ and arbitrary s. E.g. for $s = 7$ and $r = 2$ we denote a certain gamete by (A^4a^3), where $A = a_1$, $a = a_2$. The second way is to start with the fact that a gamete is given by s values,—a zygote by $2s$ values,—and to indicate these s values. This was the author's conception for arbitrary r and m, and $s = 1$ (see [4c]). We shall use whichever notation is more convenient for the respective purpose.

Obviously, (see sec. *1*) there are primarily $N' = r^{2s}$ zygotes since r figures can be arranged on $2s$ places in r^{2s} ways. But, as seen before, of these N' types many are to be considered equal to each other. As an illustration consider $s = 2$, $r = 5$ and a zygote given in the second way: $(1, 2; 5, 1) = (a_1, a_2; a_5, a_1)$. In accordance with our assumptions for $r = 2$ the following equalities must hold:

$$(1, 2; 5, 1) = (2, 1; 5, 1) = (1, 2; 1, 5) = (2, 1; 1, 5)$$
$$= (5, 1; 1, 2) = (5, 1; 2, 1) = (1, 5; 1, 2) = (1, 5; 2, 1)$$

Thus we see that *for $s = 2$ a genotype is given by two interchangeable pairs of numbers* $(x_1, x_2; y_1, y_2)$, $(x_i = 1, \cdots r, y_i = 1, \cdots r)$. There are $\binom{r}{2} + r = \binom{r+1}{2} = R$ such pairs of numbers from which we derive by combining two of them $N_1 = \frac{1}{2}R(R+1)$ different types. For $s = 2, r = 2$ we have $R = 3$, $N_1 = 6$ and indeed the six types $(A^2; A^2)$, $(A^2; Aa)$, $(A^2; a^2)$, $(Aa; Aa)$, $(Aa; a^2)$, $(a^2; a^2)$. For $s = 2, r = 5$ we have $R = 15, N_1 = 120$.—In the other conception, still for $s = 2$, a type would be given by $(a_1^{x_1}a_2^{x_2} \cdots a_r^{x_r}; a_1^{y_1}a_2^{y_2} \cdots a_r^{y_r})$ where $x_i = 0, 1, 2$, $y_i = 0, 1, 2$, $x_1 + \cdots x_r = 2, y_1 + \cdots y_r = 2$. Because of the first

of these equalities there are among the x_i either $(r-2)$ zeros and two "ones," and that can happen in $\binom{r}{2}$ ways, or $(r-1)$ zeros and one "two" and that can happen in r ways; hence again $\binom{r}{2}+r=R$ ways; now the same holds for the y_i and the expressions before and after the semicolon are interchangeable so that we find again the N_1 types.

If $s=3$ a genotype is given by two interchangeable triples and *for general s the genotype is given by two interchangeable s-tuples.* Putting $R=\binom{r+s-1}{s}$ we obviously have *for arbitrary r and s*

$$N_1=\tfrac{1}{2}R(R+1) \text{ where } R=\binom{r+s-1}{s} \tag{57}$$

types, $(x_1, \cdots x_s; y_1, \cdots y_s)$, where $x_i=1, \cdots r$, $y_i=1, \cdots r$. As an example consider, $r=8, s=5$. Here $R=792, N_1=396\cdot 793$. A certain type may be given, for instance, by:

$$(1,2,4,5,8;3,3,3,7,7), \text{ or } (a_1, a_2, a_4, a_5, a_8; a_3^3, a_7^2)$$

With the other notation a type would be denoted by $(a_1^{x_1}a_2^{x_2}\cdots a_8^{x_8}; a_1^{y_1}a_2^{y_2}\cdots a_8^{y_8})$, where $x_i=0,1,\cdots 5, y_i=0,1,\cdots 5, x_1+\cdots+x_8=5, y_1+\cdots+y_8=5$, and we have in the particular example

$$x_1=x_2=x_4=x_5=x_8=1, \qquad x_3=x_6=x_7=0,$$

$$y_1=y_2=y_4=y_5=y_6=y_8=0, \qquad y_3=3, \qquad y_7=2.$$

If we thus know how to enumerate all genotypes the *distribution of genotypes* is then defined accordingly.

Our concept of the *segregation distribution*, introduced in sec. 1, *does not change*. Still λ_i is the probability that a specified group of i maternal and a specified group of $(s-i)$ paternal genes be transmitted to the gamete, and the relations (6), (7), (8) hold.

Let us consider the *distribution of gametes*, more closely by using the "first" notation. It is, in the n^{th} generation, given as a function of s variables

$$p^{(n)}(z_1, z_2, \cdots z_s), \qquad (z_i=a_1, a_2, \cdots a_r) \tag{58}$$

Among the r^s values of (58) only $R=\binom{r+s-1}{s}$ may be different from each other, for the same reasons as before. If e.g. $r=4$, $s=8$ the probability $p(a_1^2a_2^3a_3a_4^2)$ appears $\dfrac{8!}{2!3!1!2!}=1680$ times, this being the coefficient in the polynomial expansion of

$$(a_1+a_2+a_3+a_4)^8$$

Accordingly we have, e.g. for $s=3$, omitting the upperscript $^{(n)}$:

$$p(a_1^3) + 3p(a_1^2a_2) + 3p(a_1^2a_3) + 3p(a_1a_2^2) + 3p(a_1a_3^2)$$
$$+ 3p(a_3^2a_2) + 3p(a_2^2a_3) + 6p(a_1a_2a_3) + p(a_2^3) + p(a_3^3) + \cdots = 1$$

which can be written simply

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} p(a_i a_j a_k) = 1 \tag{59}$$

Considering one more moment the case $s = 3$ we derive from this distribution the *marginal distributions of order two:*

$$s = 3: \qquad p_{12}(a_1^2) \equiv p(a_1^2) = p(a_1^3) + p(a_1^2a_2) + p(a_1^2a_3) + \cdots \text{ etc.}$$

for which

$$p(a_1^3) + p(a_2^3) + p(a_3^3) + 2p(a_1a_2) + 2p(a_1a_3) + \cdots = 1$$

and the *marginal distributions of order one*

$$s = 3: \quad p_1(a_1) \equiv p(a_1) = p(a_1^3) + p(a_1a_2^2) + p(a_1a_3^2) + 2p(a_1^2a_2) + \cdots \text{ etc.}$$

for which

$$p(a_1) + p(a_2) + \cdots + p(a_r) = 1$$

These relations may be denoted simply by

$$p_{ij}(a_ia_j) \equiv p(a_ia_j) = \sum_{k=1}^{r} p(a_ia_ja_k) \qquad \begin{pmatrix} i = 1, \cdots r \\ j = 1, \cdots r \end{pmatrix} \tag{60}$$

$$\sum_{i=1}^{r}\sum_{j=1}^{r} p(a_ia_j) = 1$$

and

$$p_i(a_i) \equiv p(a_i) = \sum_{j=1}^{r}\sum_{k=1}^{r} p(a_ia_ja_k) \qquad (i = 1, 2, \cdots r) \tag{60'}$$

$$\sum_{i=1}^{r} p(a_i) = 1$$

In an analogous way we have *for any s the marginal distribution of order j:*

$$p(a_{i_1}a_{i_2}\cdots a_{i_j}) = \sum_{i_{j+1}=1}^{r}\sum_{i_{j+2}=1}^{r}\cdots\sum_{i_s=1}^{r} p(a_{i_1}a_{i_2}\cdots a_{i_s}), \quad \begin{pmatrix} a_{i_\nu} = 1, 2, \cdots r \\ \nu = 1, 2, \cdots s \end{pmatrix} \tag{61}$$

with

$$\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\cdots\sum_{i_j=1}^{r} p(a_{i_1}a_{i_2}\cdots a_{i_j}) = 1. \tag{62}$$

8. Multiple alleles: Recurrence Formulae. Integration. Limit theorems. Let us study the *recurrence relations* for the distribution of gametes. We write again $p^{(n)} = p$, $p^{(n+1)} = p'$ and in the same way for the w-distributions. We use the notation $p(a_1^{z_1}a_2^{z_2}\cdots a_r^{z_r})$. If $r > s$ some of the z_i will necessarily be

zero. There are $\frac{s!}{z_1!\, z_2! \cdots z_r!}$ probabilities with these arguments and we first want to express $\frac{s!}{z_1!\, z_2! \cdots z_r!}\, p'(a_1^{z_1} a_2^{z_2} \cdots a_r^{z_r})$ in terms of the w'. According to the meaning of the three distributions the right side must consist of a sum of products of zygotic probabilities and segregation probabilities, each w' being multiplied by certain λ_α-values or, the other way round, each λ_α being multiplied by certain w'-values, and we want to find the coefficient of $\lambda_\alpha (\alpha = 0, 1, \cdots s)$ in this expansion. Such a λ_α is multiplied by a sum of w'-values each of which must contain α "correct" genes (or "values") before and $(s - \alpha)$ "correct" ones after the semicolon. By "α correct values" we mean α of the values which are contained in $(a_1^{z_1} a_2^{z_2} \cdots a_r^{z_r})$. Hence such a w' will be of the form

$$w'(a_1^{x_1} a_2^{x_2} \cdots a_r^{x_r} \cdots ; \quad a_1^{y_1} a_2^{y_2} \cdots a_r^{y_r} \cdots)$$

where

$$\begin{gathered} x_1 + x_2 + \cdots + x_r = \alpha, \qquad y_1 + y_2 + \cdots + y_r = s - \alpha \\ y_i + x_i = z_i\,, \quad (i = 1, 2, \cdots r), \quad z_1 + z_2 + \cdots + z_r = s \end{gathered} \tag{63}$$

In such a w' there are still $(s - \alpha)$ "free" (or "empty") places before, and α after the semicolon. Consider for the moment, to fix the ideas, a fixed choice of values for these $(s - \alpha) + \alpha$ "empty" places (put, e.g. a_1 on each of them), and let us count the w'-values after this particular choice has been decided upon: The "correct" letters before the semicolon can be arranged in $\binom{s}{\alpha}$ ways on the s places before the semicolon, and, similarly, the $(s - \alpha)$ correct ones after the semicolon in $\binom{s}{s-\alpha} = \binom{s}{\alpha}$ ways. Moreover the group of α letters $a_1^{x_1} a_2^{x_2} \cdots a_r^{x_r}$ can be arranged in $\frac{\alpha!}{x_1!\, x_2! \cdots x_r!}$ ways within their α places, and similarly the corresponding group in $\frac{(s-\alpha)!}{y_1!\, y_2! \cdots y_r!}$ ways. Hence such a w' is multiplied by

$$\binom{s}{\alpha}\binom{s}{\alpha} \frac{\alpha!}{x_1!\, x_2! \cdots x_r!} \cdot \frac{(s-\alpha)!}{y_1!\, y_2! \cdots y_r!}$$

Finally, using (13), and dividing by $\frac{s!}{z_1!\, z_2! \cdots z_r!}$ we see that the coefficient of λ_α in the expansion of $p'(a_1^{z_1} a_1^{z_2} \cdots a_r^{z_r})$ will consist of terms like

$$p(a_1^{x_1} a_2^{x_2} \cdots a_r^{x_r} \cdots) \cdot p(a_1^{y_1} a_2^{y_2} \cdots a_r^{y_r} \cdots) \binom{s}{\alpha}^2 \frac{\alpha!}{x_1! \cdots x_r!} \cdot \frac{(s-\alpha)!}{y_1! \cdots y_r!} \cdot \frac{z_1! \cdots z_r!}{s!}$$

So far we have assumed that the $(s - \alpha) + \alpha$ "free" places in a p-product are all occupied by a_1. Now we abandon this assumption. In fact, on each

of these empty places we have to write the letters $a_1, a_2, \cdots a_r$ in all r^s possible ways. This amounts to an s-tuple sum of such p-products. But this s-tuple sum of products obviously resolves in the product of the two respective $(s-\alpha)$ and α-tuple sums, the first being extended over the values on the $(s-\alpha)$ free places in the first p, the second over the values on the α free places in the second p. Now, by definition of the marginal distributions, these sums are just the marginal distributions, of order α and $(s-\alpha)$, $p(a_1^{x_1} a_2^{x_2} \cdots a_r^{x_r})$ and $p(a_1^{y_1} a_2^{y_2} \cdots a_r^{y_r})$ respectively. The coefficient of λ_α thus becomes a sum of terms like:

$$p(a_1^{x_1} \cdots a_r^{x_r}) \cdot p(a_1^{y_1} \cdots a_r^{y_r}) \binom{s}{\alpha}^2 \frac{\alpha!}{x_1! \cdots x_r!} \cdot \frac{(s-\alpha)!}{y_1! \cdots y_r!} \cdot \frac{z_1! \cdots z_r!}{s!}$$

where (61) holds for the x_i, y_i, z_i. Moreover we see easily, that

$$\frac{s!}{\alpha!(s-\alpha)!} \cdot \frac{\alpha!}{x_1! \cdots x_r!} \cdot \frac{(s-\alpha)!}{y_1! \cdots y_r!} \cdot \frac{z_1! \cdots z_r!}{s!} = \frac{z_1!}{x_1!(z_1-x_1)!} \cdots \frac{z_r!}{x_r!(z_r-x_r)!}$$

Hence the coefficient of $l_\alpha = \lambda_\alpha \binom{s}{\alpha}$ is simply

$$\sum_{z_1} \sum_{z_2} \cdots \sum_{z_r} \binom{z_1}{x_1} \cdot \binom{z_2}{x_2} \cdots \binom{z_r}{x_r}$$

and we get the elegant *general recurrence relation*

$$p'(a_1^{x_1} a_2^{x_2} \cdots a_r^{x_r}) = \sum_{\alpha=0}^{s} l_\alpha \sum_{x_1} \sum_{x_2} \cdots \sum_{x_r} \binom{z_1}{x_1}\binom{z_2}{x_2} \cdots \binom{z_r}{x_r} p(a_1^{x_1} \cdots a_r^{x_r}) p(a_1^{z_1-x_1} \cdots a_r^{z_r-x_r}) \tag{64}$$

with

$$z_1 + \cdots + z_r = s, \quad x_1 + \cdots + x_r = \alpha, \quad 0 \leqq x_i \leqq z_i, \quad (i = 1, \cdots r) \tag{64'}$$

Remark. In applications we may write in (64) ρ instead of r with the understanding that $\rho = r$ if $r \leqq s$ and $\rho = s$ if $r > s$. This rule merely means that we omit in each p these a_i which do not occur (instead of writing them with power zero.) More than s alleles can not occur in p and we may assume without loss of generality that these are "the first" s alleles.

Let us now consider *two* important *particular cases* of (64). *Firstly*, put $a_1 = A, a_2 = a, z_1 = z, z_2 = s - z$, we thus find (18) and, for $s = z$, (19). *Next* assumes $r = s, z_1 = z_2 = \cdots = z_r = 1$. We obtain the very useful formula:

$$p'(a_1 \cdots a_s) = \sum_{\alpha=0}^{s} l_\alpha [p(a_1 a_2 \cdots a_\alpha) p(a_{\alpha+1} a_{\alpha+2} \cdots a_s) + \cdots] \tag{65}$$

In each bracket there are $\binom{s}{\alpha}$ terms. If however $s = 2\mu$ the factor of l_α in the

bracket contains $\frac{1}{2}\binom{2\mu}{\mu}$ different terms only, each multiplied by two. We have e.g.

$$s = 2: \qquad p'(a_1a_2) = l_0p(a_1a_2) + 2l_1p(a_1)p(a_2) + l_2p(a_1a_2)$$

$$s = 4: \qquad p'(a_1a_2a_3a_4) = l_0p(a_1a_2a_3a_4) + l_1[p(a_1)p(a_2a_3a_4) + \cdot + \cdot + \cdot] + 2l_2[p(a_1a_2)p(a_3a_4) + p(a_1a_3)p(a_2a_4) + p(a_1a_4)p(a_2a_3)]$$

Since $l_\alpha = l_{s-\alpha}$ (65) may be written for $s = 2\mu$ or $s = 2\mu + 1$, respectively:

$$(66) \qquad p'(a_1 \cdots a_s) = \sum_{\alpha=0}^{\mu}{}' 2l_\alpha[p(a_1a_2 \cdots a_\alpha)p(a_{\alpha+1}a_{\alpha+2} \cdots a_s) + \cdots]$$

Here the dash next to the summation sign indicates that for $s = 2\mu$ the factor of l_μ contains $\frac{1}{2}\binom{2\mu}{\mu}$ terms only. Examples of (66) are

$$s = 3: \; p'(a_1a_2a_3) = 2\{l_0p(a_1a_2a_3) + l_1[p(a_1)p(a_2a_3) + \cdot + \cdot]\}$$

$$s = 4: \; p'(a_1a_2a_3a_4) = 2\{l_0p(a_1a_2a_3a_4) + l_1[p(a_1)p(a_2a_3a_4) + \cdot + \cdot + \cdot] + l_2[p(a_1a_2)p(a_3a_4) + \cdot + \cdot]\}$$

$$s = 5: \; p'(a_1a_2a_3a_4a_5) = 2\{l_0p(a_1a_2a_3a_4a_5) + l_1[p(a_1)p(a_2a_3a_4a_5) + \cdot + \cdot + \cdot + \cdot] + l_2[p(a_1a_2)p(a_3a_4a_5) + \cdots p(a_4a_5)p(a_1a_2a_3)]\}$$

By putting in (66) certain a_i equal to each other we find again (64). As an example we obtain from the last formula for $a_1 = a_4$, $a_2 = a_5$, a_3:

$$p'(a_1^2a_2^2a_3) = 2\{l_0p(a_1^2a_2^2a_3) + l_1[2p(a_1)p(a_1a_2^2a_3) + 2p(a_2)p(a_1^2a_2a_3) + \cdot] + l_2[p(a_1^2)p(a_2^2a_3) + 4p(a_1a_2)p(a_1a_2a_3) + p(a_2^2)p(a_1^2a_3) + 2p(a_1a_2)p(a_1a_2^2) + 2p(a_2a_3)p(a_1^2a_2)]\}$$

Thus it turns out that we may consider the easily remembered formulae (65) or (66) as general formulae from which any formula (64) follows by equating certain a_i to each other. (This is not true for formulae like those quoted page 259 where "chromatid segregation" is considered. There one can not obtain e.g. the formula for $p'(A^2)$ from the one for $p'(Aa)$ by putting $A = a$.)

For deriving the *limit theorem* we need *recurrence formulae for the marginal distributions*. They are of the same structure as (64) or (65), but with $l_\alpha^{(\nu)}$ instead of l_α. Consider a marginal distribution of order $(s - \nu)$. Put $\sigma = r$ if $r \leqq s - \nu$, and $\sigma = s - \nu$ if $r > s - \nu$. We then have the formula

$$(67) \qquad p'(a_1^{s_1}a_2^{s_2} \cdots a_\sigma^{s_\sigma}) = \sum_{0=\alpha}^{s-\nu} l_\alpha^{(\nu)} \sum_{x_1}\sum_{x_2} \cdots \sum_{x_\sigma} \binom{z_1}{x_1}\binom{z_2}{x_2} \cdots \binom{z_\sigma}{x_\sigma} p(a_1^{x_1} \cdots a_\sigma^{x_\sigma})p(a_1^{s_1-x_1} \cdots a_\sigma^{s_\sigma-x_\sigma})$$

where

$$(65') \qquad 0 \leqq x_i \leqq z_i\,, \qquad x_1 + \cdots + x_r = \alpha, \qquad z_1 + \cdots + z_r = s - \nu$$

and the $l_\alpha^{(\nu)}$ are the same as in (30) and (37). The analogon to (66) becomes, with $s - \nu = 2\mu$ or $2\mu + 1$ respectively

$$(68) \qquad p'(a_1 \cdots a_{s-\nu}) = 2 \sum_{\alpha=0}^{\mu}{}' \, l_\alpha^{(\nu)} [p(a_1 \cdots a_\alpha) p(a_{\alpha+1} \cdots a_{s-\nu}) + \cdots]$$

e.g. for $s = 6$, $\nu = 2$, with an obvious notation:

$$p'(a_1a_2a_3a_4) = 2\{l_0^{(2)} p_{1234} + l_1^{(2)}[p_1 p_{234} + \cdot + \cdot + \cdot] + l_2^{(2)}[p_{12}p_{34} + \cdot + \cdot]\}$$

or for $s = 6$, $\nu = 3$

$$p'(a_1a_2a_3) = 2\{l_0^{(3)} p_{123} + l_1^{(3)}[p_1 p_{23} + \cdot + \cdot]\}$$

Again we may consider (68) as the general formula from which (67) may be derived by equating certain a_i to each other. For $\nu = 0$, (67) reduces to (64), (68) to (66) and for $r = 2$ (67) becomes the last formula in sec. 4.

The proof of the *limit theorem* is not changed in any essential way by considering r alleles (see sec. 5). Our result is expressed by the simple formula; which holds *if and only if* $l_0 < \frac{1}{2}$:

$$(69) \qquad \lim_{n\to\infty} p^{(n)}(a_1^{z_1}a_2^{z_2} \cdots a_r^{z_r}) = p(a_1)^{z_1} p(a_2)^{z_2} \cdots p(a_r)^{z_r}$$

If $l_0 < \frac{1}{2}$, i.e. if "mixed gametes" are not a priori excluded, the distribution of gametes converges towards a limit where the single alleles are independently distributed. If however $l_0 = \frac{1}{2}$ the original distribution of gametes remains unaltered under random mating (see also introduction, p. 251). The limit behaviour of the distribution of genotypes follows from (69) by means of (13).

The procedure for finding an *explicit solution of the recurrence equations* is analogous to that applied for homozygotes. If we write, as before, p_i, p_{ij}, etc. for $p(a_i)$, $p(a_ia_j)$, we have for $s = 3$ e.g.

$$(70) \quad p_{123}^{(n)} = p_1p_2p_3 + (2l_0)^n[p_{123} - p_1p_2p_3]$$
$$+ [(2l_0 + 2l_1)^n - (2l_0)^n] \cdot [p_1(p_{23} - p_2p_3) + p_2(p_{13} - p_1p_3) + p_3(p_{12} - p_1p_2)]$$

This reduces to (54), (or (54')) if all alleles are equal, (or if $a_1 = a_3$, $a_2 \neq a_1$). The setup in the general case is analogous to the previous one always with the same α-values as for homozygotes. E.g.

$$p_{1234}^{(n)} = \alpha_{40}^{(n)} p_{1234} + \tfrac{1}{4}\alpha_{13}^{(n)}[p_{123}p_4 + p_{124}p_3 + p_{134}p_2 + p_{234}p_1] + \alpha_{22}^{(n)} \cdot \tfrac{1}{3}[p_{12}p_{34} + \cdot + \cdot]$$
$$+ \alpha_{211}^{(n)} \cdot \tfrac{1}{6}[p_{12}p_3p_4 + \cdots + p_{34}p_1p_2] + \alpha_{1111}^{(n)} p_1p_2p_3p_4$$

with the α-values (sec. 6)

$$\alpha_{40}^{(n)} = (2l_0)^n, \qquad \tfrac{1}{4}\alpha_{13}^{(n)} = (2l_0 + 2l_1)^n - (2l_0)^n, \cdots$$

Thus we state that *in the integration of the general recurrence formulae* (64) *the same α-coefficients appear as in the integration problem of sec.* 6, *for homozygotes,*

i.e. as in the case when all a_i equal each other. The only difference is that a product like $p(A^i)p(A^j)\cdots$ is now to be replaced by the sum of all possible combinations which can be formed in analogy to this product by means of several alleles, and it is to be divided by the number of all such combinations (see the above examples.) As a further example consider in the expression for $p^{(n)}(A^6)$ the term $a^{(n)}_{321}p(A^3)p(A^2)p(A)$. This will be replaced by:

$$\tfrac{1}{6!}a^{(n)}_{321}[p_{123}(p_{45}p_6 + p_{46}p_5 + p_{56}p_4) + \cdots + p_{456}(p_{12}p_3 + p_{13}p_2 + p_{23}p_1)]$$

There are in fact 60 terms in this bracket. Thus we may obtain any desired explicit solution.—Our results concerning the *rate of approach to equilibrium* remain, of course, unchanged in case of multiple alleles.

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NUMERICAL SOLUTION OF COMPLEX ROOTS OF QUARTIC EQUATIONS

By Shih-Nge Lin

Reference 1 (Shih-Nge Lin, A Method for Finding Roots of Algebraic Equations. Journal of Mathematics and Physics, 23, 60–77, 1943) described a method for finding roots of algebraic equations. The following description offers an improvement on the solution of complex roots of quartic equations having double or nearly double quadratic factors.

Consider the quartic equation:

$$x^4 + A_3x^3 + A_2x^2 + A_1x + A_0 = 0,$$

which can be factored into $(x^2 + b_1x + b_0)(x^2 + a_1x + a_0) = 0$.
We have,

$$A_1 = A_3 - b_1 \quad (1), \qquad a_1b_1 = A_2 - (a_0 + b_0) \quad (2),$$

$$A_1 - a_1b_0 - a_0b_1 = 0 \quad (3), \qquad a_0b_0 = A_0 \quad (4).$$

Eliminating a_1 and a_0 by (1), (3) and (4),

$$b_0 = \frac{A_1}{2(A_3 - b_1)} \pm \sqrt{\left[\frac{A_1}{2(A_3 - b_1)}\right]^2 - \frac{A_0b_1}{(A_3 - b_1)}} \qquad (A).$$

Eliminating a_1 and a_0 by (1), (2) and (4),

$$b_0 = \frac{A_2 - b_1(A_3 - b_1)}{2} \pm \sqrt{\left(\frac{A_2 - b_1(A_3 - b_1)}{2}\right)^2 - A_0} \qquad (B).$$

It is seen that the necessary conditions for the constant terms a_0 and b_0 of the quadratic factors to be equal are that the second terms of (A) and (B) should be zero, namely

$$b_1 - A_3b_1 + A_1^2/4A_0 = 0 \qquad (A)'$$

and

$$b_1 - A_3b_1 + (A_2 - 2\sqrt{A_0}) = 0 \qquad (B)'.$$

Equations (A)′ and (B)′ offer relations in which values of a_1 and b_1 may be approximated when a_0 and b_0 are of values close to each other. To illustrate this possibility, let $\alpha = A_1/2\sqrt{A_0}$, $\beta = A_2 - 2\sqrt{A_0}$, $\gamma = A_3/2$, then, we shall have following cases:

Case (I) $\beta = \alpha^2 = \gamma^2$. The given quartic equation is seen to have identical pairs of quadratic factors, and $a_0 = b_0 = +\sqrt{A_0}$, $a_1 = b_1 = A_3/2$.

Case (II) $\beta < \alpha^2 = \gamma^2$. In this case, the constant terms, a_0 and b_0 of the quadratic factors of the given equations are seen to be equal and the quartic can be exactly factored by determining a_1 and b_1 as real roots of $(B)'$, giving

$$a_1, b_1 = \gamma \pm \sqrt{\gamma^2 - \beta} \qquad (C).$$

(Note that if $\alpha^2 = \gamma^2$, it can be shown that $a_0 = b_0 = \sqrt{A_0}$ as follows: Eliminating b_0 in (3) and (4), we get

$$a_0 = \frac{A_1}{2b_1} \pm \sqrt{\left(\frac{A_1}{2b_1}\right)^2 - \left(\frac{a_1}{b_1}\right) A_0} = \frac{\sqrt{A_0}}{b_1}\left[\alpha \pm \sqrt{\alpha^2 - a_1 b_1}\right].$$

But by (C), $a_1 b_1 = \beta$, thus,

$$a_0 = \frac{\sqrt{A_0}}{(\gamma \pm \sqrt{\gamma^2 - \beta})}\,(\alpha \pm \sqrt{\alpha^2 - \beta}) = \sqrt{A_0}\,).$$

Consider the following numerical example:

$$x^4 + 4.45x^3 + 15.1x^2 + 24.9x + 31.36 = 0.$$

We have:

$$\alpha = \frac{24.9}{2\sqrt{31.36}} = 2.225,\ \beta = 15.1 - 2\sqrt{31.36} = 3.90,\ \gamma = 4.45/2 = 2.225.$$

Thus, by (C),

$$a_1\,, b_1 = 2.225 \pm \sqrt{(2.225)^2 - 3.90} = 3.25, 1.20,$$

and

$$a_0\,, b_0 = \sqrt{31.36} = 5.60.$$

The given quartic is then factored into

$$(x^2 + 3.25x + 5.60)(x^2 + 1.20x + 5.60) = 0.$$

Case (III) $\beta < \alpha^2 \neq \gamma^2$. In this case, the given equation can be conveniently solved by the following iteration procedure:

Step (1), determine the first approximations for a_1 , b_1 by (C).

Step (2), determine the first approximations for a_0 , b_0 by

$$b_0 = (\sqrt{A_0}/a_1)(\alpha \pm \sqrt{\alpha^2 - \beta}), \tag{D}$$

which is obtained by eliminating a_0 from (3) and (4). And by (4), we have

$$a_0 = A_0/b_0 \tag{E}.$$

It is seen that the first approximations shall satisfy (1), (3) and (4), but in general not (2).

Step (3), Calculate β' such that

$$\beta' = A_2 - (a_0 - b_0) \tag{F}.$$

If β' is equal to β, then (2) is also satisfied, and the first approximations are exact solutions. If not, calculate the second approximations for a_1 , b_1 , a_0 and b_0 from (C), (D) and (E) by using β' instead of β. The process is thus repeated to the nth number of times until

$$B'_n = A_2 - (a_0 + b_0)_n = \beta_n = (a_1 b_1)_n \qquad \text{(G)}.$$

In general, convergency of the process is so rapid that engineering accuracy can be reached within four or five repetitions. When (G) is satisfied, (2) is also satisfied so that the quartic equation is exactly factored.

Case (IV). When β is greater than either one or both of α^2 and γ^2, the quadratic factors of the given equation can be considered to be "separated" so that method described in reference (1) can be advantageously applied with good convergence.

Numerical illustrations:

Example (1)

$$x^4 - 5x^3 + 12.5x^2 - 16.5x + 10.5 = 0$$

The characteristic parameters are found to be

$$\alpha = -2.545, \alpha^2 = 6.48, \beta = 6.02, \gamma = -2.50, \gamma^2 = 6.25.$$

Thus, we have case (III). For first approximations, from (C), we get

$$a_1 = -2.5 + \sqrt{6.25 - 6.02} = -2.02$$

and

$$b_1 = -2.5 - \sqrt{6.25 - 6.02} = -2.98.$$

From (D) and (E), we have

$$b_0 = \frac{\sqrt{10.5}}{-2.02}(-2.545 + \sqrt{6.48 - 6.02}) = 2.99, a_0 = 10.5/2.99 = 3.51.$$

It is seen that

$$\beta' = 12.5 - (3.51 + 2.99) = 6.00,$$

which is very close to 6.02. For second approximations, substitute 6.00 instead of 6.02 into (C) and (D), we get

$$a_1 = -2.50 + \sqrt{6.25 - 6.00} = -2.00; b_1 = -2.5 - \sqrt{6.25 - 6.00} = -3.00$$

$$b_0 = \frac{\sqrt{10.5}}{-2.00}(-2.545 + (\sqrt{6.48 - 6.00}) = 3.00; a_0 = 10.5/3.00 = 3.50.$$

It is seen that

$$\beta' = 12.5 - (3.50 + 3.00) = 6.00 = (-2.00)(-3.00) = \beta.$$

Thus, the given equation can be considered to be factored into

$$(x^2 - 2.00x + 3.50)(x^2 - 3.00x + 3.00) = 0.$$

Example (2)

$$x - 2.075x^3 + 0.69x^2 - 0.095x + 2.287 = 0.$$

The test parameters are

$$\alpha = -.0314, \alpha^2 = 0.000986; \beta = -2.355, \gamma = -1.0375,$$

and $\gamma^2 = 1.075$. Thus, we have also case (III). Hence, for first approximation, $a_1, b_1 = -1.0375 \pm \sqrt{1.075 + 2.355} = -2.8925, 0.8175; b_0 = (1.515/-2.8925) \times (-.0314 - \sqrt{.000986 + 2.355}) = 0.822$. $a_0 = 2.287/0.822 = 2.78$.

The first approximation gives

$$\beta' = 0.67 - (0.822 + 2.78) = -2.932.$$

By repeating the process, we get the following:

APPROXIMATIONS	β	a_1	b_1	b_0	a_0
first	−2.355	−2.8925	0.8175	0.822	2.78
second	−2.932	−3.0375	0.9625	0.870	2.63
third	−2.830	−3.0135	0.9385	0.863	2.65
fourth	−2.843	−3.0155	0.9405	0.863	2.65
fifth	−2.843				

Thus, the given equation is factored into

$$(x^2 - 3.0155x + 2.65)(x^2 + 0.9405x + 0.863) = 0.$$

Example (3)

$$x^4 + 5.31x^3 + 6.52x^2 + 0.71x + 0.24 = 0.$$

The test parameters are

$$\beta = 5.54, \alpha^2 = 0.527, \gamma^2 = 7.06.$$

Thus, we have case (IV), and the quartic equation is found to be easily factored by the method of reference (1), giving

$$(x^2 + 0.08485x + 0.03975)(x^2 + 5.22515x + 6.037) = 0.$$

Example (4)

$$x^4 - 6x^3 + 47x^2 - 18x + 290 = 0.$$

The test parameters are

$$\beta = 12.94, \alpha^2 = 0.279, \gamma^2 = 9.0.$$

Thus, we have also case (IV) and the equation has been easily solved in reference (1).

Conclusion. The improvement here in combination with the method of reference 1 offers a convenient method for the solution of quartic equations with two

pairs of complex roots. The coefficients A_2 and A_0 of the quartic equation having all roots complex are always positive so that the method is applicable. When either one of the coefficients A_2 and A_0 is not positive, it is readily known that real roots are present and the equation can usually be solved by Newton's method.

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ON THE STRESSES IN A NOTCHED PLATE UNDER TENSION

CHIH-BING LING

The problem of a notched plate under tension, when the plate is of semi-infinite size and the notch is in semicircular form, was solved by Maunsell [1]. It belongs to a class of plate problems in which the boundary consists of intersecting parts of different nature. By the use of bipolar coordinates, it appears that the solution can be extended to cover any circular notch in general (or mound, as a negative notch) without being restricted to semicircular. Such a solution will be presented in this paper.

The biharmonic equation is first solved by means of infinite integrals. A suitable even solution is then used to construct the required stress function. The parametric functions involved in the solution are subsequently adjusted to satisfy the boundary conditions of the notched plate.

It may be mentioned that, besides the problem of notched plate, the integral solution in general can be used to solve two additional plate problems, namely: (1) a crescent, or a plate bounded by two circular arcs, and (2) an infinite plate containing a crescent-shaped hole. It is thus interesting to compare the integral solution with the series solution obtained by Jeffery, which is also in bipolar coordinates [2]. Jeffery's solution can be used to solve three analogous plate problems, namely: (1) an infinite plate containing two circular holes of any radii, (2) a semi-infinite plate bounded by a straight edge and containing a circular hole and (3) a circular disc containing an eccentric circular hole.

The bipolar coordinates (ξ, η) will be defined by

$$x + iy = -a \coth \tfrac{1}{2}i(\xi + i\eta) \tag{1}$$

such that the two poles of the coordinates are on the x-axis at the points $(\pm a, 0)$, and

$$x = J \sinh \eta, \qquad y = J \sin \xi$$

where

$$a/J = \cosh \eta - \cos \xi \tag{2}$$

The biharmonic equation $\nabla^4\chi = 0$ which must be satisfied by the stress function χ then transforms to

$$\left(\frac{\partial^4}{\partial\xi^4} + 2\,\frac{\partial^4}{\partial\xi^2\partial\eta^2} + \frac{\partial^4}{\partial\eta^4} + 2\,\frac{\partial^2}{\partial\xi^2} - 2\,\frac{\partial^2}{\partial\eta^2} + 1\right)\frac{\chi}{J} = 0 \tag{3}$$

Consider a solution even in η of the type $\chi/J = F_n(\xi)\cos n\eta$. The transformed equation reduces on substitution to

$$\{d^4/d\xi^4 - 2(n^2 - 1)\,d^2/d\xi^2 + (n^2 + 1)^2\}\,F_n(\xi) = 0 \tag{4}$$

The solution of this equation is readily found to be

$$F_n(\xi) = (f_n \sin \xi + h_n \cos \xi)\sinh n\xi + (g_n \sin \xi + k_n \cos \xi)\cosh n\xi \tag{5}$$

where f_n, h_n, $\cdots$ are parametric functions of n. Provided that the integral converges, a more general solution even in η is therefore

$$\chi_1/J = \int_0^\infty F_n(\xi) \cos n\eta \, dn \tag{6}$$

A similar solution but odd in η is as follows:

$$\chi_1^*/J = \int_0^\infty F_n^*(\xi) \sin n\eta \, dn \tag{7}$$

where

$$F_n^*(\xi) = (f_n^* \sin \xi + h_n^* \cos \xi) \sinh n\xi + (g_n^* \sin \xi + k_n^* \cos \xi) \cosh n\xi$$

A general solution may therefore be obtained by combining (6) and (7) together with some function relating to some particular stress system. The parametric functions involved are then adjusted to satisfy the boundary conditions of the problem. For the notched plate here under consideration, the boundary conditions are as follows (see Fig. 1): (1) at infinity $x = \pm\infty$ or $y = \infty$; or $\xi, \eta = 0$

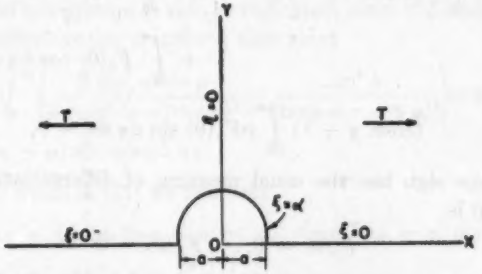

FIG. 1. A NOTCHED PLATE UNDER TENSION

$$\widehat{xx} = T, \text{ say}, \qquad \widehat{yy} = 0, \qquad \widehat{yx} = 0 \tag{8}$$

(2) along the edge $y = 0$, save the part $-a \leq x \leq a$; or $\xi = 0$

$$\widehat{\xi\xi} = 0, \qquad \widehat{\eta\xi} = 0 \tag{9}$$

and (3) along the rim of the notch ($\alpha < \pi$) or mound ($\pi < \alpha$) $\xi = \alpha$,

$$\widehat{\xi\xi} = 0, \qquad \widehat{\xi\eta} = 0 \tag{10}$$

Now, in an unnotched plate a uniform longitudinal tension T would be given by the stress function

$$\chi_0 = \tfrac{1}{2}Ty^2, \text{ or } \qquad \chi_0/J = \tfrac{1}{2}aT \sin^2 \xi/(\cosh \eta - \cos \xi) \tag{11}$$

The method of solution when the notch is present is to construct the required stress function χ (which must be even in η by symmetry) in the form:

$$\chi = \chi_0 + aT\chi_1 \tag{12}$$

where χ_1 is the biharmonic solution even in η as shown in (6); the factor aT being introduced to render the parametric functions involved nondimensional.

The stresses in bipolar coordinates derived from a stress function χ are equal to [3]

$$a\widehat{\xi\xi} = \{(\cosh\eta - \cos\xi)\partial^2/\partial\eta^2 - \sinh\eta\,\partial/\partial\eta - \sin\xi\,\partial/\partial\xi + \cos\xi\}(\chi/J)$$

$$a\widehat{\eta\eta} = \{(\cosh\eta - \cos\xi)\partial^2/\partial\xi^2 - \sinh\eta\,\partial/\partial\eta - \sin\xi\,\partial/\partial\xi + \cosh\eta\}(\chi/J)$$

$$a\widehat{\xi\eta} = -(\cosh\eta - \cos\xi)\partial^2(\chi/J)/\partial\xi\partial\eta \tag{13}$$

The boundary conditions at infinity thus require that

$$\int_0^\infty F_n(0)\,dn = 0 \tag{14}$$

Those at the edge $\xi = 0$ require that

$$\left.\begin{aligned} -(\cosh\eta - 1)\int_0^\infty n^2F_n(0)\cos n\eta\,dn + \sinh\eta\int_0^\infty nF_n(0)\sin n\eta\,dn \\ + \int_0^\infty F_n'(0)\cos n\eta\,dn = 0, \\ (\cosh\eta - 1)\int_0^\infty nF_n'(0)\sin n\eta\,dn = 0, \end{aligned}\right\} \tag{15}$$

where the prime sign has the usual meaning of differentiation. An obvious solution of (15) is

$$F_n(0) = 0, \qquad F_n'(0) = 0 \tag{16}$$

Here $F_n(0) = 0$ is only a particular solution of the first equation, but it also satisfies the condition at infinity. The solution thus gives

$$k_n = 0, \qquad g_n = -nh_n \tag{17}$$

Again, the boundary conditions at the rim $\xi = \alpha$ require that

$$\left.\begin{aligned} \sin^2\alpha\sinh^2\eta/(\cosh\eta - \cos\alpha)^2 - (\cosh\eta - \cos\alpha)\int_0^\infty n^2F_n(\alpha)\cos n\eta\,dn \\ + \sinh\eta\int_0^\infty nF_n(\alpha)\sin n\eta\,dn + \cos\alpha\int_0^\infty F_n(\alpha)\cos n\eta\,dn \\ - \sin\alpha\int_0^\infty F_n'(\alpha)\cos n\eta\,dn = 0, \\ -\sin\alpha\sinh\eta(1 - \cos\alpha\cosh\eta)/(\cosh\eta - \cos\alpha)^2 \\ + (\cosh\eta - \cos\alpha)\int_0^\infty nF_n'(\alpha)\sin n\eta\,dn = 0 \end{aligned}\right\} \tag{18}$$

Here the second equation is satisfied if

$$\int_0^\infty nF_n'(\alpha)\sin n\eta\,dn = \sin\alpha\sinh\eta(1 - \cos\alpha\cosh\eta)/(\cosh\eta - \cos\alpha)^3 \tag{19}$$

Fourier sine transform gives

$$nF_n'(\alpha) = \frac{2\sin\alpha}{\pi}\int_0^\infty \frac{(1-\cos\alpha\cosh\eta)\sinh\eta\sin n\eta\,d\eta}{(\cosh\eta-\cos\alpha)^3} \tag{20}$$

By Cauchy's theorem of residues, we obtain without difficulty.

$$F_n'(\alpha)\sinh n\pi = n\cosh n(\pi-\alpha)\sin\alpha - \sinh n(\pi-\alpha)\cos\alpha \tag{21}$$

To determine $F_n(\alpha)$, we first differentiate the first equation with respect to η. Then with the aid of the second equation, we find

$$\int_0^\infty n(n^2+1)F_n(\alpha)\sin n\eta\,dn = -\frac{3\sin^2\alpha\sinh\eta\,(1-\cos\alpha\cosh\eta)}{(\cosh\eta-\cos\alpha)^4} \tag{22}$$

Integration with respect to η yields

$$\int_0^\infty (n^2+1)F_n(\alpha)\cos n\eta\,dn = \frac{3\sin^2\alpha\cos\alpha}{2(\cosh\eta-\cos\alpha)^2} - \frac{\sin^4\alpha}{(\cosh\eta-\cos\alpha)^3} \tag{23}$$

Here the constant of integration is zero as indicated, since the integral vanishes when $\eta=\infty$. Fourier's cosine transform then gives

$$\begin{aligned} F_n(\alpha) &= \frac{2}{(n^2+1)\pi}\int_0^\infty\left\{\frac{3\sin^2\alpha\cos\alpha}{2(\cosh\eta-\cos\alpha)^2} - \frac{\sin^4\alpha}{(\cosh\eta-\cos\alpha)^3}\right\}\cos n\eta\,d\eta \\ &= -\sinh n(\pi-\alpha)\sin\alpha/\sinh n\pi. \end{aligned} \tag{24}$$

Combining with the result in (21) we thus find

$$\begin{aligned} f_n &= \coth n\pi - \tfrac{1}{2}(\sinh 2n\alpha - n\sin 2\alpha)/(\sinh^2 n\alpha - n^2\sin^2\alpha) \\ h_n &= -n\sin^2\alpha/(\sinh^2 n\alpha - n^2\sin^2\alpha) \end{aligned} \tag{25}$$

To summarize, the required stress function χ is given by

$$\chi/aTJ = \tfrac{1}{2}\sin^2\xi/(\cosh\eta-\cos\xi) + \int_0^\infty F_n(\xi)\cos n\eta\,dn \tag{26}$$

where, with f_n and h_n in (25),

$$F_n(\xi) = f_n\sinh n\xi\sin\xi - h_n(n\cosh n\xi\sin\xi - \sinh n\xi\cos\xi) \tag{27}$$

It is now a simple matter to find the stress at any point in the plate. In particular, it is found that the stress at the rim $\xi=\alpha$ is given by

$$\frac{[\widehat{\eta\eta}]_a}{T} = 2(\cosh\eta-\cos\alpha)\int_0^\infty \frac{n(n\cosh n\alpha\sin\alpha - \sinh n\alpha\cos\alpha)}{\sinh^2 n\alpha - n^2\sin^2\alpha}\cos n\eta\,dn \tag{28}$$

It possesses a stationary value when $\eta=0$, i.e. at the 'crown', which is a maximum in a notch and a minimum in a mound. By putting

$$\theta = n\alpha, \qquad c = \sin\alpha/\alpha, \qquad b = \alpha\cot\alpha \tag{29}$$

the 'crown stress' is expressed by

$$\frac{[\widehat{\eta\eta}]_c}{T} = \frac{2(1-\cos\alpha)\sin\alpha}{\alpha^3}\int_0^\infty \frac{\theta(\theta\cosh\theta - b\sinh\theta)\,d\theta}{\sinh^2\theta - c^2\theta^2} \tag{30}$$

This form is more suitable for numerical integration. The value thus expressed is usually called 'stress concentration factor'. In the limiting case $\alpha = 0$ it becomes

$$\frac{[\widehat{\eta\eta}]_c}{T} = \int_0^\infty \frac{\theta(\theta \cosh \theta - \sinh \theta)\, d\theta}{\sinh^2 \theta - \theta^2} \tag{31}$$

This limiting case refers in fact to a plate pierced by a circular hole tangential to the edge. The numerical values of the factor are shown in the following table

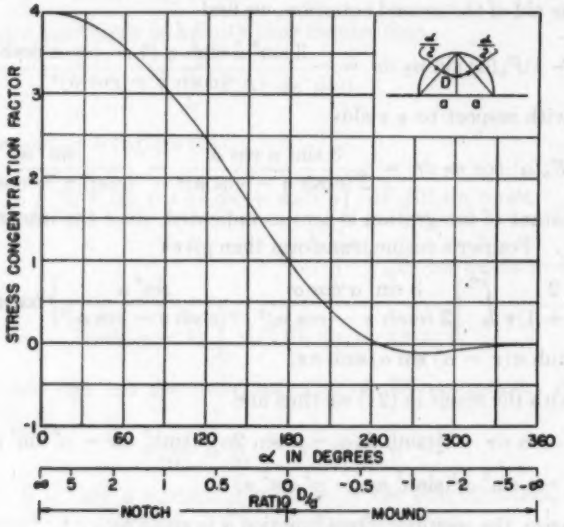

FIG. 2. STRESS CONCENTRATION FACTOR

and also shown graphically in Fig. 2; D being the distance from the crown to the edge.*

* I am indebted to a referee for drawing my attention to some earlier works on the problem by Weinel [4] and by Isibasi [5]. The present analysis was carried out before Weinel's paper became known in this country. Isabasi's work was mentioned in a footnote by Weinel and was unfortunately not available to me.

Weinel's paper is so concise that only the final results were given. His treatment was slightly different from the present one. However, identical expressions were found for the maximum stress. A comparison of the numerical values may be made as follows:

α	0	90°	180°
present.....	3.999	3.065	1.000
Weinel's.....	3.990	3.063	1.000

Weinel did not discuss the problem of the mound.

α	D/a	S. C. FACTOR	REMARK
0	∞	3.999	Tangential hole
30°	3.732	3.882	
60°	1.732	3.568	
90°	1.000	3.065	Semicircular notch
120°	0.577	2.424	
150°	0.268	1.707	
180°	0.000	1.000	Unnotched plate
210°	−0.268	0.411	
240°	−0.577	0.039	
270°	−1.000	−0.096	Semicircular mound
300°	−1.732	−0.076	
330°	−3.732	−0.021	
360°	$-\infty$	−0.000	Tangential mound

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NOTE ON THE MEMBRANE THEORY OF SHELLS OF REVOLUTION

By Eric Reissner

1. Introduction and Formulation of the Problem. The treatment of problems without axial symmetry in the linear membrane theory of shells, which starts with the work of H. Reissner [1] who initiated this theory and gave the solution for the spherical shell, has recently been advanced by the introduction of stress functions. In terms of such a stress function the problem can be reduced to one second order linear partial differential equation. Two such stress functions have been proposed, one by A. Pucher [2], [3], [4] and one by P. Nemenyi [5], [6]. The connection between these two kinds of stress functions has as yet not been discussed in the literature. The present note has as its purpose to establish this connection and to offer a slight generalization of Nemenyi's stress function.

Omitting, as is often appropriate in membrane theory, consideration of displacements, the problem consists in the integration of three stress resultant equilibrium equations. For a shell of revolution these equations may be written in the form

$$\partial r N_r/\partial \xi + r' \partial N_{\xi\theta}/\partial \theta - \alpha N_\theta + r\alpha p_r = 0 \tag{1}$$

$$\partial r N_{\xi\theta}/\partial \xi + \alpha\, \partial N_\theta/\partial \theta + r' N_{\xi\theta} + r\alpha p_\theta = 0 \tag{2}$$

$$\partial r \tan\phi\, N_r/\partial \xi + z'\, \partial N_{\xi\theta}/\partial \theta + r\alpha p_z = 0. \tag{3}$$

In equations (1) to (3) $N_r = N_\xi \cos\phi$, and N_ξ, N_θ and $N_{\xi\theta}$ are the stress resultants. The meridian curve of the shell is taken in the form $r = r(\xi)$, $z = z(\xi)$, the angle ϕ is defined by $\tan\phi = dz/dr$, and $\alpha = [(r')^2 + (z')^2]^{1/2}$. Primes indicate differentiation with respect to ξ.

2. Pucher's stress function. It may be noted that by writing $\partial/\partial\xi = r'\,\partial/\partial r$ equations (1) and (2) for N_r, $N_{\xi\theta}$ and $(\alpha/r')N_\theta$ assume the same form as the equations of plane stress for σ_r, $\tau_{r\theta}$ and σ_θ. Consequently, (1) and (2) are satisfied in terms of a stress function F as follows

$$(\alpha/r')N_\theta = \partial^2 F/\partial r^2 + A \tag{4a}$$

$$N_r = r^{-1}\,\partial F/\partial r + r^{-2}\,\partial^2 F/\partial\theta^2 + B \tag{4b}$$

$$N_{\xi\theta} = -\partial(r^{-1}\,\partial F/\partial\theta)/\partial r. \tag{4c}$$

The terms A and B are given in terms of the external loads in the form

$$r' A = -r\alpha \int p_\theta\, d\theta, \qquad rB = \int \{A - r\alpha p_r/r'\}\, dr. \tag{5}$$

Substitution of (4) into (3) gives as differential equation for F,

$$\frac{dz}{dr}\frac{\partial^3 F}{\partial r^3} + \frac{d^2 z}{dr^2}\left(\frac{\partial F}{\partial r} + \frac{1}{r}\frac{\partial^2 F}{\partial\theta^2}\right) + \frac{r\alpha}{r'} p_z + \frac{\partial r z' B}{\partial r} = 0. \tag{6}$$

For the purpose of comparison with Nemenyi's results we note that equation (6), without the external load terms, may be written in the form

$$\partial^2 F/\partial z^2 - z(f''/f')\partial F/\partial z - (f''/f)\partial^2 F/\partial\theta^2 = 0 \tag{7}$$

where now the meridian curve is given by an equation $r = f(z)$. If we further introduce a function F^* defined by

$$F = f'F^* \tag{8}$$

we have an equation containing no first derivatives of F^* of the form

$$\partial^2 F^*/\partial z^2 + [(f'''/f') - 2(f''/f')^2]F^* - (f''/f)\partial^2 F^*/\partial\theta^2 = 0. \tag{9}$$

3. Nemenyi's stress function. A consideration of Nemenyi's results suggests to write, in terms of a stress function U^*, for the solution of the equations without the surface load terms,

$$N_r = g_1\partial U^*/\partial\theta, \qquad N_\theta = g_2\,\partial U^*/\partial\theta, \qquad N_{\xi\theta} = g_3\,\partial U^*/\partial\xi. \tag{10}$$

In (10) the factors g_1, g_2 and g_3 are functions of ξ which must be determined suitably. Introducing (10) into (1) to (3) there follows:

$$[(rg_1)' - \alpha g_2]\partial U^*/\partial\theta + [rg_1 + r'g_3]\partial^2 U^*/\partial\xi\partial\theta = 0 \tag{11}$$

$$[(rg_3)' + r'g_3]\partial U^*/\partial\xi + rg_3\partial^2 U^*/\partial\xi^2 + \alpha g_2\,\partial^2 U^*/\partial\theta^2 = 0 \tag{12}$$

$$[(r\tan\phi\ g_1)']\partial U^*/\partial\theta + [r\tan\phi\ g_1 + z'g_3]\partial^2 U^*/\partial\xi\partial\theta = 0. \tag{13}$$

Noting that the second bracket in (13) equals the second bracket in (11) multiplied by the factor $\tan\phi = z'/r'$, the crucial fact is observed that (11) and (13) can both be satisfied identically by a proper choice of the functions g_n. We get

$$(rg_1)' - \alpha g_2 = 0, \qquad rg_1 + r'g_3 = 0, \qquad (rz'g_1/r')' = 0 \tag{14}$$

and obtain

$$g_1 = r'/rz', \qquad g_2 = (r'/z')'/\alpha, \qquad g_3 = -1/z'. \tag{15}$$

Substituting these functions g_n into the remaining equation (12) there follows as equation for U^*,

$$\boxed{\frac{\partial^2 U^*}{\partial\xi^2} + \left[\frac{(r/z')'}{r/z'} + \frac{r'}{r}\right]\frac{\partial U^*}{\partial\xi} - \frac{(r'/z')'}{r/z'}\frac{\partial^2 U^*}{\partial\theta^2} = 0} \tag{16}$$

Combining (10) and (15) we have for the stress resultants in terms of U^*

$$\boxed{N_r = \frac{r'}{rz'}\frac{\partial U^*}{\partial\theta}, \qquad N_T = \frac{(r'/z')'}{\sqrt{(r')^2 + (z')^2}}\frac{\partial U^*}{\partial\theta}, \qquad N_{\xi\theta} = -\frac{1}{z'}\frac{\partial U^*}{\partial\xi}} \tag{17}$$

To obtain Nemenyi's result we identify the parameter ξ with the coordinate z, so that $z' = 1$, and write $r = f(z)$. Writing further

$$U = fU^* \tag{18}$$

in order to eliminate first derivative terms we obtain

$$\boxed{\frac{\partial^2 U}{\partial z} - \frac{f''(z)}{f(z)}\left(U + \frac{\partial^2 U}{\partial \theta^2}\right) = 0} \tag{19}$$

When $U = U_n(z) \cos n\theta$ equation (19) reduces to Nemenyi's equation for U_n.

With U^* from (18) and with $\xi = z$ equations (10) and (15) lead to the following expressions for the stress resultants

$$\boxed{N_r = \frac{f'}{f^2}\frac{\partial U}{\partial \theta}, \qquad N_\theta = \frac{f''}{\sqrt{1 + (f')^2}}\frac{\partial U}{\partial \theta}, \qquad N_{\xi\theta} = -\frac{\partial}{\partial z}\left(\frac{U}{f}\right)} \tag{20}$$

In comparing (19) and (20) with the corresponding results (7) or (9) and (4) it appears that the use of the stress function U is preferable to the use of F, because the differential equation for U is simpler than that for F or F^* and because first derivatives only of U are needed for the determination of the stresses while second derivatives are needed of F for the same purpose.

It is interesting to note that with the use of F the conditions of radial and circumferential equilibrium are satisfied identically and the condition of axial equilibrium is employed to determine F, while with the use of U the conditions of radial and axial equilibrium are satisfied identically while the condition of circumferential equilibrium is employed to determine U.

We may remark that equation (19) includes the solution for the case of an axially symmetric stress distribution, in the form $U = c\theta f(z)$. To the stresses corresponding to this solution must be added the particular integrals of (1) to (3) which take account of surface loads.*

4. Relation between U and F. It suggests itself to derive a relation showing the connection between the two stress functions U and F, particularly in view of a remark made by Truesdell [7, p. 128].

If we compare the last of equations (20), with the last of equations (4) we see that

$$\partial(U/f)/\partial z = (1/f')\partial(f^{-1}\partial F/\partial\theta)/\partial z. \tag{21}$$

It is sufficient to compare $U = U_n \cos n\,\theta$ with $F = F_n \sin n\,\theta$. Then

$$(U_n/f)' = (n/f')(F_n/f)' \tag{22}$$

and thus

$$\boxed{F_n = (f/n)\int (U_n/f)' f'\, dz.} \tag{23}$$

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* To take care of surface loads one adds in (20) the particular integrals

$$N_{r,p} = -r^{-1}(dr/dz)\int r\alpha p_z\, d\xi, \quad N_{\xi,p} = 0, \quad N_{\theta,p} = rp_r + \alpha^{-1}\partial(rN_{r,p})/\partial\xi$$

where upon the righthand sides of the differential equations for U* and for U assume values different from zero [6].

This then is the relation between the function F_n obtained by Pucher and the function U_n obtained by Nemenyi. It may readily be shown that equation (23) which holds whether or not surface loads are present is also in agreement with the two sets of expressions for the remaining stress resultants N_r and N_θ.

One might think that cases may occur for which the function U_n is an elementary or tabulated function while F_n is not. That this cannot be so may be shown as follows. Integrate (23) by parts and use equation (19) as reduced for U_n. There follows, except for the cases $n = 0$ and $n = 1$ which are readily treated separately, and assuming for simplicity sake the absence of surface loads,

$$F_n = (f'/n)U_n + [f/n(n^2 - 1)]U_n'. \tag{24}$$

We should add that Nemenyi and Truesdell have given *explicit* solutions for many interesting special cases, whereas Pucher's applications depend mainly on the possibility of obtaining power-series solutions for his function F_n.

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TABLES OF COEFFICIENTS FOR INTERPOLATING IN FUNCTIONS OF TWO VARIABLES

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In a previous note*, a theorem was established concerning the multiple Gregory-Newton interpolation formula (see Appendix I (1')), which is applicable to tabulated functions of any number of variables. That theorem showed how to rearrange (1') so as to avoid the computation of any of the partial differences. Because of its general interest, another alternative proof of that theorem is provided in Appendix I. (This alternative proof is the author's original one, whereas the previously published proof was essentially due to Professor W. E. Milne.)

The most important case is where the interpolation is in a function-table of two variables. Then (1') becomes

$$(1) \qquad f(x + ph_1, y + qh_2) = \sum_{i+j=0}^{n} \binom{p}{i}\binom{q}{j} \Delta_x^{i+j}{}_{i,j} f(x, y),$$

where h_1, h_2 denote the tabular intervals, $\binom{p}{i} = \dfrac{p(p-1)\cdots(p-i+1)}{i!}$, with $\binom{p}{0} = 1$, and $\Delta_x^{i+j}{}_{i,j} f(x, y)$ denotes the mixed partial advancing difference of $f(x, y)$ of order i with respect to x and order j with respect to y, and the summation is for all sets of values of i, j such that $i + j$ goes from 0 to n. By that general theorem given below, (1) can be written as

$$(2) \qquad f(x + ph_1, y + qh_2) = \sum_{s+t=0}^{n} C_{s,t} f_{s,t} = \sum_{s+t=0}^{n} \binom{n-p-q}{n-s-t}\binom{p}{s}\binom{q}{t} f_{s,t},$$

where s and t assume all values such that $s + t$ ranges from 0 to n.

The double Gregory-Newton formula is useful in a wide variety of cases where the range of interpolation permits $f(x, y)$ to be approximated by a binary n-ic. An important application is whenever Taylor's theorem for two variables holds in a suitable neighborhood of the point $f(x_0, y_0)$, up to the same total degree n. To see how close an approximation may be had by Taylor's expansion, one must usually know the values of the various mixed partial derivatives of order $n + 1$, or some other means of estimating the error. Of course, the error in approximating $f(x, y)$ by a Taylor expansion around $f(x_0, y_0)$ merely indicates the order n of the interpolation formula to be employed; and the user must carefully distinguish between that error which serves as a guide in choosing the n, and the actual error incurred in the use of the nth order double Gregory-Newton interpolation formula, namely,

$$(3) \qquad R_n = \sum \left[\frac{\partial^{n+1} f(x, y)}{\partial x^i \, \partial y^j}\right]_{\substack{x=\xi \\ y=\eta}} h_1^i h_2^j \binom{p}{i}\binom{q}{j},$$

* H. E. Salzer, "Note on Interpolation for a Function of Several Variables", Bull. Am. Math. Soc., 51, 1945, pp. 279–280.

TABLES OF COEFFICIENTS

Quadratic Interpolation

q = .1

p	A_{00}	A_{01}	A_{02}	A_{10}	A_{11}	A_{20}
.1	.720	.180	−.045	.180	.010	−.045
.2	.595	.170	−.045	.340	.020	−.080
.3	.480	.160	−.045	.480	.030	−.105
.4	.375	.150	−.045	.600	.040	−.120
.5	.280	.140	−.045	.700	.050	−.125
.6	.195	.130	−.045	.780	.060	−.120
.7	.120	.120	−.045	.840	.070	−.105
.8	.055	.110	−.045	.880	.080	−.080
.9	.000	.100	−.045	.900	.090	−.045

q = .2

p	A_{00}	A_{01}	A_{02}	A_{10}	A_{11}	A_{20}
.1	.595	.340	−.080	.170	.020	−.045
.2	.480	.320	−.080	.320	.040	−.080
.3	.375	.300	−.080	.450	.060	−.105
.4	.280	.280	−.080	.560	.080	−.120
.5	.195	.260	−.080	.650	.100	−.125
.6	.120	.240	−.080	.720	.120	−.120
.7	.055	.220	−.080	.770	.140	−.105
.8	.000	.200	−.080	.800	.160	−.080
.9	−.045	.180	−.080	.810	.180	−.045

q = .3

p	A_{00}	A_{01}	A_{02}	A_{10}	A_{11}	A_{20}
.1	.480	.480	−.105	.160	.030	−.045
.2	.375	.450	−.105	.300	.060	−.080
.3	.280	.420	−.105	.420	.090	−.105
.4	.195	.390	−.105	.520	.120	−.120
.5	.120	.360	−.105	.600	.150	−.125
.6	.055	.330	−.105	.660	.180	−.120
.7	.000	.300	−.105	.700	.210	−.105
.8	−.045	.270	−.105	.720	.240	−.080
.9	−.080	.240	−.105	.720	.270	−.045

q = .4

p	A_{00}	A_{01}	A_{02}	A_{10}	A_{11}	A_{20}
.1	.375	.600	−.120	.150	.040	−.045
.2	.280	.560	−.120	.280	.080	−.080
.3	.195	.520	−.120	.390	.120	−.105
.4	.120	.480	−.120	.480	.160	−.120
.5	.055	.440	−.120	.550	.200	−.125
.6	.000	.400	−.120	.600	.240	−.120
.7	−.045	.360	−.120	.630	.280	−.105
.8	−.080	.320	−.120	.640	.320	−.080
.9	−.105	.280	−.120	.630	.360	−.045

q = .5

p	A_{00}	A_{01}	A_{02}	A_{10}	A_{11}	A_{20}
.1	.280	.700	−.125	.140	.050	−.045
.2	.195	.650	−.125	.260	.100	−.080
.3	.120	.600	−.125	.360	.150	−.105
.4	.055	.550	−.125	.440	.200	−.120
.5	.000	.500	−.125	.500	.250	−.125
.6	−.045	.450	−.125	.540	.300	−.120
.7	−.080	.400	−.125	.560	.350	−.105
.8	−.105	.350	−.125	.560	.400	−.080
.9	−.120	.300	−.125	.540	.450	−.045

q = .6

p	A_{00}	A_{01}	A_{02}	A_{10}	A_{11}	A_{20}
.1	.195	.780	−.120	.130	.060	−.045
.2	.120	.720	−.120	.240	.120	−.080
.3	.055	.660	−.120	.330	.180	−.105
.4	.000	.600	−.120	.400	.240	−.120
.5	−.045	.540	−.120	.450	.300	−.125
.6	−.080	.480	−.120	.480	.360	−.120
.7	−.105	.420	−.120	.490	.420	−.105
.8	−.120	.360	−.120	.480	.480	−.080
.9	−.125	.300	−.120	.450	.540	−.045

q = .7

p	A_{00}	A_{01}	A_{02}	A_{10}	A_{11}	A_{20}
.1	.120	.840	−.105	.120	.070	−.045
.2	.055	.770	−.105	.220	.140	−.080
.3	.000	.700	−.105	.300	.210	−.105
.4	−.045	.630	−.105	.360	.280	−.120
.5	−.080	.560	−.105	.400	.350	−.125
.6	−.105	.490	−.105	.420	.420	−.120
.7	−.120	.420	−.105	.420	.490	−.105
.8	−.125	.350	−.105	.400	.560	−.080
.9	−.120	.280	−.105	.360	.630	−.045

q = .8

p	A_{00}	A_{01}	A_{02}	A_{10}	A_{11}	A_{20}
.1	.055	.880	−.080	.110	.080	−.045
.2	.000	.800	−.080	.200	.160	−.080
.3	−.045	.720	−.080	.270	.240	−.105
.4	−.080	.640	−.080	.320	.320	−.120
.5	−.105	.560	−.080	.350	.400	−.125
.6	−.120	.480	−.080	.360	.480	−.120
.7	−.125	.400	−.080	.350	.560	−.105
.8	−.120	.320	−.080	.320	.640	−.080
.9	−.105	.240	−.080	.270	.720	−.045

q = .9

p	A_{00}	A_{01}	A_{02}	A_{10}	A_{11}	A_{20}
.1	.000	.900	−.045	.100	.090	−.045
.2	−.045	.810	−.045	.180	.180	−.080
.3	−.080	.720	−.045	.240	.270	−.105
.4	−.105	.630	−.045	.280	.360	−.120
.5	−.120	.540	−.045	.300	.450	−.125
.6	−.125	.450	−.045	.300	.540	−.120
.7	−.120	.360	−.045	.280	.630	−.105
.8	−.105	.270	−.045	.240	.720	−.080
.9	−.080	.180	−.045	.180	.810	−.045

where in the summation $i + j$ ranges over all values $\geqslant 0$ whose sum is $n + 1$, and ξ, η are certain mean values of x, y in the given right-triangle region. Since

Cubic Interpolation

q = .1

p	B_{00}	B_{01}	B_{02}	B_{03}	B_{10}	B_{11}	B_{12}	B_{30}	B_{21}	B_{30}	p
.1	.6720	.2520	−.1260	.0285	.2520	.0280	−.0045	−.1260	−.0045	.0285	.1
.2	.5355	.2295	−.1215	.0285	.4590	.0540	−.0090	−.2160	−.0080	.0480	.2
.3	.4160	.2080	−.1170	.0285	.6240	.0780	−.0135	−.2730	−.0105	.0595	.3
.4	.3125	.1875	−.1125	.0285	.7500	.1000	−.0180	−.3000	−.0120	.0640	.4
.5	.2240	.1680	−.1080	.0285	.8400	.1200	−.0225	−.3000	−.0125	.0625	.5
.6	.1495	.1495	−.1035	.0285	.8970	.1380	−.0270	−.2760	−.0120	.0560	.6
.7	.0880	.1320	−.0990	.0285	.9240	.1540	−.0315	−.2310	−.0105	.0455	.7
.8	.0385	.1155	−.0945	.0285	.9240	.1680	−.0360	−.1680	−.0080	.0320	.8
.9	.0000	.1000	−.0900	.0285	.9000	.1800	−.0405	−.0900	−.0045	.0165	.9

q = .2

p	B_{00}	B_{01}	B_{02}	B_{03}	B_{10}	B_{11}	B_{12}	B_{30}	B_{21}	B_{30}	p
.1	.5355	.4590	−.2160	.0480	.2295	.0540	−.0080	−.1215	−.0090	.0285	.1
.2	.4160	.4160	−.2080	.0480	.4160	.1040	−.0160	−.2080	−.0160	.0480	.2
.3	.3125	.3750	−.2000	.0480	.5625	.1500	−.0240	−.2625	−.0210	.0595	.3
.4	.2240	.3360	−.1920	.0480	.6720	.1920	−.0320	−.2880	−.0240	.0640	.4
.5	.1495	.2990	−.1840	.0480	.7475	.2300	−.0400	−.2875	−.0250	.0625	.5
.6	.0880	.2640	−.1760	.0480	.7920	.2640	−.0480	−.2640	−.0240	.0560	.6
.7	.0385	.2310	−.1680	.0480	.8085	.2940	−.0560	−.2205	−.0210	.0455	.7
.8	.0000	.2000	−.1600	.0480	.8000	.3200	−.0640	−.1600	−.0160	.0320	.8
.9	−.0285	.1710	−.1520	.0480	.7695	.3420	−.0720	−.0855	−.0090	.0165	.9

q = .3

p	B_{00}	B_{01}	B_{02}	B_{03}	B_{10}	B_{11}	B_{12}	B_{30}	B_{21}	B_{30}	p
.1	.4160	.6240	−.2730	.0595	.2080	.0780	−.0105	−.1170	−.0135	.0285	.1
.2	.3125	.5625	−.2625	.0595	.3750	.1500	−.0210	−.2000	−.0240	.0480	.2
.3	.2240	.5040	−.2520	.0595	.5040	.2160	−.0315	−.2520	−.0315	.0595	.3
.4	.1495	.4485	−.2415	.0595	.5980	.2760	−.0420	−.2760	−.0360	.0640	.4
.5	.0880	.3960	−.2310	.0595	.6600	.3300	−.0525	−.2750	−.0375	.0625	.5
.6	.0385	.3465	−.2205	.0595	.6930	.3780	−.0630	−.2520	−.0360	.0560	.6
.7	.0000	.3000	−.2100	.0595	.7000	.4200	−.0735	−.2100	−.0315	.0455	.7
.8	−.0285	.2565	−.1995	.0595	.6840	.4560	−.0840	−.1520	−.0240	.0320	.8
.9	−.0480	.2160	−.1890	.0595	.6480	.4860	−.0945	−.0810	−.0135	.0165	.9

q = .4

p	B_{00}	B_{01}	B_{02}	B_{03}	B_{10}	B_{11}	B_{12}	B_{30}	B_{21}	B_{30}	p
.1	.3125	.7500	−.3000	.0640	.1875	.1000	−.0120	−.1125	−.0180	.0285	.1
.2	.2240	.6720	−.2880	.0640	.3360	.1920	−.0240	−.1920	−.0320	.0480	.2
.3	.1495	.5980	−.2760	.0640	.4485	.2760	−.0360	−.2415	−.0420	.0595	.3
.4	.0880	.5280	−.2640	.0640	.5280	.3520	−.0480	−.2640	−.0480	.0640	.4
.5	.0385	.4620	−.2520	.0640	.5775	.4200	−.0600	−.2625	−.0500	.0625	.5
.6	.0000	.4000	−.2400	.0640	.6000	.4800	−.0720	−.2400	−.0480	.0560	.6
.7	−.0285	.3420	−.2280	.0640	.5985	.5320	−.0840	−.1995	−.0420	.0455	.7
.8	−.0480	.2880	−.2160	.0640	.5760	.5760	−.0960	−.1440	−.0320	.0320	.8
.9	−.0595	.2380	−.2040	.0640	.5355	.6120	−.1080	−.0765	−.0180	.0165	.9

R_n is approximately the sum of all the $(n + 1)$th order terms in the double Gregory-Newton formula, in actual practise an estimate of the error in using the

q = .5

p	B_{00}	B_{01}	B_{02}	B_{03}	B_{10}	B_{11}	B_{12}	B_{20}	B_{21}	B_{30}	p
.1	.2240	.8400	−.3000	.0625	.1680	.1200	−.0125	−.1080	−.0225	.0285	.1
.2	.1495	.7475	−.2875	.0625	.2990	.2300	−.0250	−.1840	−.0400	.0480	.2
.3	.0880	.6600	−.2750	.0625	.3960	.3300	−.0375	−.2310	−.0525	.0595	.3
.4	.0385	.5775	−.2625	.0625	.4620	.4200	−.0500	−.2520	−.0600	.0640	.4
.5	.0000	.5000	−.2500	.0625	.5000	.5000	−.0625	−.2500	−.0625	.0625	.5
.6	−.0285	.4275	−.2375	.0625	.5130	.5700	−.0750	−.2280	−.0600	.0560	.6
.7	−.0480	.3600	−.2250	.0625	.5040	.6300	−.0875	−.1890	−.0525	.0455	.7
.8	−.0595	.2975	−.2125	.0625	.4760	.6800	−.1000	−.1360	−.0400	.0320	.8
.9	−.0640	.2400	−.2000	.0625	.4320	.7200	−.1125	−.0720	−.0225	.0165	.9

q = .6

p	B_{00}	B_{01}	B_{02}	B_{03}	B_{10}	B_{11}	B_{12}	B_{20}	B_{21}	B_{30}	p
.1	.1495	.8970	−.2760	.0560	.1495	.1380	−.0120	−.1035	−.0270	.0285	.1
.2	.0880	.7920	−.2640	.0560	.2640	.2640	−.0240	−.1760	−.0480	.0480	.2
.3	.0385	.6930	−.2520	.0560	.3465	.3780	−.0360	−.2205	−.0630	.0595	.3
.4	.0000	.6000	−.2400	.0560	.4000	.4800	−.0480	−.2400	−.0720	.0640	.4
.5	−.0285	.5130	−.2280	.0560	.4275	.5700	−.0600	−.2375	−.0750	.0625	.5
.6	−.0480	.4320	−.2160	.0560	.4320	.6480	−.0720	−.2160	−.0720	.0560	.6
.7	−.0595	.3570	−.2040	.0560	.4165	.7140	−.0840	−.1785	−.0630	.0455	.7
.8	−.0640	.2880	−.1920	.0560	.3840	.7680	−.0960	−.1280	−.0480	.0320	.8
.9	−.0625	.2250	−.1800	.0560	.3375	.8100	−.1080	−.0675	−.0270	.0165	.9

q = .7

p	B_{00}	B_{01}	B_{02}	B_{03}	B_{10}	B_{11}	B_{12}	B_{20}	B_{21}	B_{30}	p
.1	.0880	.9240	−.2310	.0455	.1320	.1540	−.0105	−.0990	−.0315	.0285	.1
.2	.0385	.8085	−.2205	.0455	.2310	.2940	−.0210	−.1680	−.0560	.0480	.2
.3	.0000	.7000	−.2100	.0455	.3000	.4200	−.0315	−.2100	−.0735	.0595	.3
.4	−.0285	.5985	−.1995	.0455	.3420	.5320	−.0420	−.2280	−.0840	.0640	.4
.5	−.0480	.5040	−.1890	.0455	.3600	.6300	−.0525	−.2250	−.0875	.0625	.5
.6	−.0595	.4165	−.1785	.0455	.3570	.7140	−.0630	−.2040	−.0840	.0560	.6
.7	−.0640	.3360	−.1680	.0455	.3360	.7840	−.0735	−.1680	−.0735	.0455	.7
.8	−.0625	.2625	−.1575	.0455	.3000	.8400	−.0840	−.1200	−.0560	.0320	.8
.9	−.0560	.1960	−.1470	.0455	.2520	.8820	−.0945	−.0630	−.0315	.0165	.9

q = .8

p	B_{00}	B_{01}	B_{02}	B_{03}	B_{10}	B_{11}	B_{12}	B_{20}	B_{21}	B_{30}	p
.1	.0385	.9240	−.1680	.0320	.1155	.1680	−.0080	−.0945	−.0360	.0285	.1
.2	.0000	.8000	−.1600	.0320	.2000	.3200	−.0160	−.1600	−.0640	.0480	.2
.3	−.0285	.6840	−.1520	.0320	.2565	.4560	−.0240	−.1995	−.0840	.0595	.3
.4	−.0480	.5760	−.1440	.0320	.2880	.5760	−.0320	−.2160	−.0960	.0640	.4
.5	−.0595	.4760	−.1360	.0320	.2975	.6800	−.0400	−.2125	−.1000	.0625	.5
.6	−.0640	.3840	−.1280	.0320	.2880	.7680	−.0480	−.1920	−.0960	.0560	.6
.7	−.0625	.3000	−.1200	.0320	.2625	.8400	−.0560	−.1575	−.0840	.0455	.7
.8	−.0560	.2240	−.1120	.0320	.2240	.8960	−.0640	−.1120	−.0640	.0320	.8
.9	−.0455	.1560	−.1040	.0320	.1755	.9360	−.0720	−.0585	−.0360	.0165	.9

q = .9

p	B_{00}	B_{01}	B_{02}	B_{03}	B_{10}	B_{11}	B_{12}	B_{20}	B_{21}	B_{30}	p
.1	.0000	.9000	−.0900	.0165	.1000	.1800	−.0045	−.0900	−.0405	.0285	.1
.2	−.0285	.7695	−.0855	.0165	.1710	.3420	−.0090	−.1520	−.0720	.0480	.2
.3	−.0480	.6480	−.0810	.0165	.2160	.4860	−.0135	−.1890	−.0945	.0595	.3
.4	−.0595	.5355	−.0765	.0165	.2380	.6120	−.0180	−.2040	−.1080	.0640	.4
.5	−.0640	.4320	−.0720	.0165	.2400	.7200	−.0225	−.2000	−.1125	.0625	.5
.6	−.0625	.3375	−.0675	.0165	.2250	.8100	−.0270	−.1800	−.1080	.0560	.6
.7	−.0560	.2520	−.0630	.0165	.1960	.8820	−.0315	−.1470	−.0945	.0455	.7
.8	−.0455	.1755	−.0585	.0165	.1560	.9360	−.0360	−.1040	−.0720	.0320	.8
.9	−.0320	.1080	−.0540	.0165	.1080	.9720	−.0405	−.0540	−.0405	.0165	.9

Quartic Interpolation

q = .1

p	C_{20}	C_{21}	C_{22}	C_{23}	C_{24}	C_{10}	C_{11}
.1	.6384000	.3192000	−.2394000	.1083000	−.0206625	.3192000	.0532000
.2	.4953375	.2830500	−.2247750	.1054500	−.0206625	.5661000	.0999000
.3	.3744000	.2496000	−.2106000	.1026000	−.0206625	.7488000	.1404000
.4	.2734375	.2187500	−.1968750	.0997500	−.0206625	.8750000	.1750000
.5	.1904000	.1904000	−.1836000	.0969000	−.0206625	.9520000	.2040000
.6	.1233375	.1644500	−.1707750	.0940500	−.0206625	.9867000	.2277000
.7	.0704000	.1408000	−.1584000	.0912000	−.0206625	.9856000	.2464000
.8	.0298375	.1193500	−.1464750	.0883500	−.0206625	.9548000	.2604000
.9	.0000000	.1000000	−.1350000	.0855000	−.0206625	.9000000	.2700000

q = .2

p	C_{20}	C_{21}	C_{22}	C_{23}	C_{24}	C_{10}	C_{11}
.1	.4953375	.5661000	−.3996000	.1776000	−.0336000	.2830500	.0999000
.2	.3744000	.4992000	−.3744000	.1728000	−.0336000	.4992000	.1872000
.3	.2734375	.4375000	−.3500000	.1680000	−.0336000	.6562500	.2625000
.4	.1904000	.3808000	−.3264000	.1632000	−.0336000	.7616000	.3264000
.5	.1233375	.3289000	−.3036000	.1584000	−.0336000	.8222500	.3795000
.6	.0704000	.2816000	−.2816000	.1536000	−.0336000	.8448000	.4224000
.7	.0298375	.2387000	−.2604000	.1488000	−.0336000	.8354500	.4557000
.8	.0000000	.2000000	−.2400000	.1440000	−.0336000	.8000000	.4800000
.9	−.0206625	.1653000	−.2204000	.1392000	−.0336000	.7438500	.4959000

q = .3

p	C_{20}	C_{21}	C_{22}	C_{23}	C_{24}	C_{10}	C_{11}
.1	.3744000	.7488000	−.4914000	.2142000	−.0401625	.2496000	.1404000
.2	.2734375	.6562500	−.4593750	.2082500	−.0401625	.4375000	.2625000
.3	.1904000	.5712000	−.4284000	.2023000	−.0401625	.5712000	.3672000
.4	.1233375	.4933500	−.3984750	.1963500	−.0401625	.6578000	.4554000
.5	.0704000	.4224000	−.3696000	.1904000	−.0401625	.7040000	.5280000
.6	.0298375	.3580500	−.3417750	.1844500	−.0401625	.7161000	.5859000
.7	.0000000	.3000000	−.3150000	.1785000	−.0401625	.7000000	.6300000
.8	−.0206625	.2479500	−.2892750	.1725500	−.0401625	.6612000	.6612000
.9	−.0336000	.2016000	−.2646000	.1666000	−.0401625	.6048000	.6804000

q = .4

p	C_{20}	C_{21}	C_{22}	C_{23}	C_{24}	C_{10}	C_{11}
.1	.2734375	.8750000	−.5250000	.2240000	−.0416000	.2187500	.1750000
.2	.1904000	.7616000	−.4896000	.2176000	−.0416000	.3808000	.3264000
.3	.1233375	.6578000	−.4554000	.2112000	−.0416000	.4933500	.4554000
.4	.0704000	.5632000	−.4224000	.2048000	−.0416000	.5632000	.5632000
.5	.0298375	.4774000	−.3906000	.1984000	−.0416000	.5967500	.6510000
.6	.0000000	.4000000	−.3600000	.1920000	−.0416000	.6000000	.7200000
.7	−.0206625	.3306000	−.3306000	.1856000	−.0416000	.5785500	.7714000
.8	−.0336000	.2688000	−.3024000	.1792000	−.0416000	.5376000	.8064000
.9	−.0401625	.2142000	−.2754000	.1728000	−.0416000	.4819500	.8262000

nth order formula can be had by using the $(n + 1)$th order formula and noting the difference. A few random tests furnish an adequate basis for estimating the error for a large number of examples in a region.

Another widespread application of the double Gregory-Newton formula is in

the separate interpolation for the real or imaginary part of an analytic function of total degree n. In such cases, where n is often determinable from the Taylor

Quartic Interpolation

q = .1

C_{22}	C_{12}	C_{02}	C_{21}	C_{11}	C_{01}	C_{20}	C_{10}	C_{00}	p
−.0171000	.0028500	−.2394000	−.0171000	.0020250	.1083000	.0028500	−.0206625		.1
−.0333000	.0057000	−.3996000	−.0296000	.0036000	.1776000	.0048000	−.0336000		.2
−.0486000	.0085500	−.4914000	−.0378000	.0047250	.2142000	.0059500	−.0401625		.3
−.0630000	.0114000	−.5250000	−.0420000	.0054000	.2240000	.0064000	−.0416000		.4
−.0765000	.0142500	−.5100000	−.0425000	.0056250	.2125000	.0062500	−.0390625		.5
−.0891000	.0171000	−.4554000	−.0396000	.0054000	.1848000	.0056000	−.0336000		.6
−.1008000	.0199500	−.3696000	−.0336000	.0047250	.1456000	.0045500	−.0261625		.7
−.1116000	.0228000	−.2604000	−.0248000	.0036000	.0992000	.0032000	−.0176000		.8
−.1215000	.0256500	−.1350000	−.0135000	.0020250	.0495000	.0016500	−.0086625		.9

q = .2

C_{22}	C_{12}	C_{02}	C_{21}	C_{11}	C_{01}	C_{20}	C_{10}	C_{00}	p
−.0296000	.0048000	−.2247750	−.0333000	.0036000	.1054500	.0057000	−.0206625		.1
−.0576000	.0096000	−.3744000	−.0576000	.0064000	.1728000	.0096000	−.0336000		.2
−.0840000	.0144000	−.4593750	−.0735000	.0084000	.2082500	.0119000	−.0401625		.3
−.1088000	.0192000	−.4896000	−.0816000	.0096000	.2176000	.0128000	−.0416000		.4
−.1320000	.0240000	−.4743750	−.0825000	.0100000	.2062500	.0125000	−.0390625		.5
−.1536000	.0288000	−.4224000	−.0768000	.0096000	.1792000	.0112000	−.0336000		.6
−.1736000	.0336000	−.3417750	−.0651000	.0084000	.1410500	.0091000	−.0261625		.7
−.1920000	.0384000	−.2400000	−.0480000	.0064000	.0960000	.0064000	−.0176000		.8
−.2088000	.0432000	−.1239750	−.0261000	.0036000	.0478500	.0033000	−.0086625		.9

q = .3

C_{22}	C_{12}	C_{02}	C_{21}	C_{11}	C_{01}	C_{20}	C_{10}	C_{00}	p
−.0378000	.0059500	−.2106000	−.0486000	.0047250	.1026000	.0085500	−.0206625		.1
−.0735000	.0119000	−.3500000	−.0840000	.0084000	.1680000	.0144000	−.0336000		.2
−.1071000	.0178500	−.4284000	−.1071000	.0110250	.2023000	.0178500	−.0401625		.3
−.1386000	.0238000	−.4554000	−.1188000	.0126000	.2112000	.0192000	−.0416000		.4
−.1680000	.0297500	−.4400000	−.1200000	.0131250	.2000000	.0187500	−.0390625		.5
−.1953000	.0357000	−.3906000	−.1116000	.0126000	.1736000	.0168000	−.0336000		.6
−.2205000	.0416500	−.3150000	−.0945000	.0110250	.1365000	.0136500	−.0261625		.7
−.2436000	.0476000	−.2204000	−.0696000	.0084000	.0928000	.0096000	−.0176000		.8
−.2646000	.0535500	−.1134000	−.0378000	.0047250	.0462000	.0049500	−.0086625		.9

q = .4

C_{22}	C_{12}	C_{02}	C_{21}	C_{11}	C_{01}	C_{20}	C_{10}	C_{00}	p
−.0420000	.0064000	−.1968750	−.0630000	.0054000	.0997500	.0114000	−.0206625		.1
−.0816000	.0128000	−.3264000	−.1088000	.0096000	.1632000	.0192000	−.0336000		.2
−.1188000	.0192000	−.3984750	−.1386000	.0126000	.1963500	.0238000	−.0401625		.3
−.1536000	.0256000	−.4224000	−.1536000	.0144000	.2048000	.0256000	−.0416000		.4
−.1860000	.0320000	−.4068750	−.1550000	.0150000	.1937500	.0250000	−.0390625		.5
−.2160000	.0384000	−.3600000	−.1440000	.0144000	.1680000	.0224000	−.0336000		.6
−.2436000	.0448000	−.2892750	−.1218000	.0126000	.1319500	.0182000	−.0261625		.7
−.2688000	.0512000	−.2016000	−.0896000	.0096000	.0896000	.0128000	−.0176000		.8
−.2916000	.0576000	−.1032750	−.0486000	.0054000	.0445500	.0066000	−.0086625		.9

series around $f(z_0) \equiv f(x_0 + iy_0)$, one has exactly a binary form of degree n, to which the double Gregory-Newton formula is applicable, and for which it would

q = .5

p	C_{00}	C_{01}	C_{02}	C_{03}	C_{04}	C_{10}	C_{11}
.1	.1904000	.9520000	−.5100000	.2125000	−.0390625	.1904000	.2040000
.2	.1233375	.8222500	−.4743750	.2062500	−.0390625	.3289000	.3795000
.3	.0704000	.7040000	−.4400000	.2000000	−.0390625	.4224000	.5280000
.4	.0298375	.5967500	−.4068750	.1937500	−.0390625	.4774000	.6510000
.5	.0000000	.5000000	−.3750000	.1875000	−.0390625	.5000000	.7500000
.6	−.0206625	.4132500	−.3443750	.1812500	−.0390625	.4959000	.8265000
.7	−.0336000	.3360000	−.3150000	.1750000	−.0390625	.4704000	.8820000
.8	−.0401625	.2677500	−.2868750	.1687500	−.0390625	.4284000	.9180000
.9	−.0416000	.2080000	−.2600000	.1625000	−.0390625	.3744000	.9360000

q = .6

p	C_{00}	C_{01}	C_{02}	C_{03}	C_{04}	C_{10}	C_{11}
.1	.1233375	.9867000	−.4554000	.1848000	−.0336000	.1644500	.2277000
.2	.0704000	.8448000	−.4224000	.1792000	−.0336000	.2816000	.4224000
.3	.0298375	.7161000	−.3906000	.1736000	−.0336000	.3580500	.5859000
.4	.0000000	.6000000	−.3600000	.1680000	−.0336000	.4000000	.7200000
.5	−.0206625	.4959000	−.3306000	.1624000	−.0336000	.4132500	.8265000
.6	−.0336000	.4032000	−.3024000	.1568000	−.0336000	.4032000	.9072000
.7	−.0401625	.3213000	−.2754000	.1512000	−.0336000	.3748500	.9639000
.8	−.0416000	.2496000	−.2496000	.1456000	−.0336000	.3328000	.9984000
.9	−.0390625	.1875000	−.2250000	.1400000	−.0336000	.2812500	1.0125000

q = .7

p	C_{00}	C_{01}	C_{02}	C_{03}	C_{04}	C_{10}	C_{11}
.1	.0704000	.9856000	−.3696000	.1456000	−.0261625	.1408000	.2464000
.2	.0298375	.8354500	−.3417750	.1410500	−.0261625	.2387000	.4557000
.3	.0000000	.7000000	−.3150000	.1365000	−.0261625	.3000000	.6300000
.4	−.0206625	.5785500	−.2892750	.1319500	−.0261625	.3306000	.7714000
.5	−.0336000	.4704000	−.2646000	.1274000	−.0261625	.3360000	.8820000
.6	−.0401625	.3748500	−.2409750	.1228500	−.0261625	.3213000	.9639000
.7	−.0416000	.2912000	−.2184000	.1183000	−.0261625	.2912000	1.0192000
.8	−.0390625	.2187500	−.1968750	.1137500	−.0261625	.2500000	1.0500000
.9	−.0336000	.1568000	−.1764000	.1092000	−.0261625	.2016000	1.0584000

q = .8

p	C_{00}	C_{01}	C_{02}	C_{03}	C_{04}	C_{10}	C_{11}
.1	.0298375	.9548000	−.2604000	.0992000	−.0176000	.1193500	.2604000
.2	.0000000	.8000000	−.2400000	.0960000	−.0176000	.2000000	.4800000
.3	−.0206625	.6612000	−.2204000	.0928000	−.0176000	.2479500	.6612000
.4	−.0336000	.5376000	−.2016000	.0896000	−.0176000	.2688000	.8064000
.5	−.0401625	.4284000	−.1836000	.0864000	−.0176000	.2677500	.9180000
.6	−.0416000	.3328000	−.1664000	.0832000	−.0176000	.2496000	.9984000
.7	−.0390625	.2500000	−.1500000	.0800000	−.0176000	.2187500	1.0500000
.8	−.0336000	.1792000	−.1344000	.0768000	−.0176000	.1792000	1.0752000
.9	−.0261625	.1196000	−.1196000	.0736000	−.0176000	.1345500	1.0764000

q = .9

p	C_{00}	C_{01}	C_{02}	C_{03}	C_{04}	C_{10}	C_{11}
.1	.0000000	.9000000	−.1350000	.0495000	−.0086625	.1000000	.2700000
.2	−.0206625	.7438500	−.1239750	.0478500	−.0086625	.1653000	.4959000
.3	−.0336000	.6048000	−.1134000	.0462000	−.0086625	.2016000	.6804000
.4	−.0401625	.4819500	−.1032750	.0445500	−.0086625	.2142000	.8262000
.5	−.0416000	.3744000	−.0936000	.0429000	−.0086625	.2080000	.9360000
.6	−.0390625	.2812500	−.0843750	.0412500	−.0086625	.1875000	1.0125000
.7	−.0336000	.2016000	−.0756000	.0396000	−.0086625	.1568000	1.0584000
.8	−.0261625	.1345500	−.0672750	.0379500	−.0086625	.1196000	1.0764000
.9	−.0176000	.0792000	−.0594000	.0363000	−.0086625	.0792000	1.0692000

q = .5

C_{12}	C_{13}	C_{20}	C_{21}	C_{22}	C_{23}	C_{31}	C_{40}	p
−.0425000	.0062500	−.1836000	−.0765000	.0056250	.0969000	.0142500	−.0206625	.1
−.0825000	.0125000	−.3036000	−.1320000	.0100000	.1584000	.0240000	−.0336000	.2
−.1200000	.0187500	−.3696000	−.1680000	.0131250	.1904000	.0297500	−.0401625	.3
−.1550000	.0250000	−.3906000	−.1860000	.0150000	.1984000	.0320000	−.0416000	.4
−.1875000	.0312500	−.3750000	−.1875000	.0156250	.1875000	.0312500	−.0390625	.5
−.2175000	.0375000	−.3306000	−.1740000	.0150000	.1624000	.0280000	−.0336000	.6
−.2450000	.0437500	−.2646000	−.1470000	.0131250	.1274000	.0227500	−.0261625	.7
−.2700000	.0500000	−.1836000	−.1080000	.0100000	.0864000	.0160000	−.0176000	.8
−.2925000	.0562500	−.0936000	−.0585000	.0056250	.0429000	.0082500	−.0086625	.9

q = .6

C_{12}	C_{13}	C_{20}	C_{21}	C_{22}	C_{23}	C_{31}	C_{4}	p
−.0396000	.0056000	−.1707750	−.0891000	.0054000	.0940500	.0171000	−.0206625	.1
−.0768000	.0112000	−.2816000	−.1536000	.0096000	.1536000	.0288000	−.0336000	.2
−.1116000	.0168000	−.3417750	−.1953000	.0126000	.1844500	.0357000	−.0401625	.3
−.1440000	.0224000	−.3600000	−.2160000	.0144000	.1920000	.0384000	−.0416000	.4
−.1740000	.0280000	−.3443750	−.2175000	.0150000	.1812500	.0375000	−.0390625	.5
−.2016000	.0336000	−.3024000	−.2016000	.0144000	.1568000	.0336000	−.0336000	.6
−.2268000	.0392000	−.2409750	−.1701000	.0126000	.1228500	.0273000	−.0261625	.7
−.2496000	.0448000	−.1664000	−.1248000	.0096000	.0832000	.0192000	−.0176000	.8
−.2700000	.0504000	−.0843750	−.0675000	.0054000	.0412500	.0099000	−.0086625	.9

q = .7

C_{12}	C_{13}	C_{20}	C_{21}	C_{22}	C_{23}	C_{31}	C_{40}	p
−.0336000	.0045500	−.1584000	−.1008000	.0047250	.0912000	.0199500	−.0206625	.1
−.0651000	.0091000	−.2604000	−.1736000	.0084000	.1488000	.0336000	−.0336000	.2
−.0945000	.0136500	−.3150000	−.2205000	.0110250	.1785000	.0416500	−.0401625	.3
−.1218000	.0182000	−.3306000	−.2436000	.0126000	.1856000	.0448000	−.0416000	.4
−.1470000	.0227500	−.3150000	−.2450000	.0131250	.1750000	.0437500	−.0390625	.5
−.1701000	.0273000	−.2754000	−.2268000	.0126000	.1512000	.0392000	−.0336000	.6
−.1911000	.0318500	−.2184000	−.1911000	.0110250	.1183000	.0318500	−.0261625	.7
−.2100000	.0364000	−.1500000	−.1400000	.0084000	.0800000	.0224000	−.0176000	.8
−.2268000	.0409500	−.0756000	−.0756000	.0047250	.0396000	.0115500	−.0086625	.9

q = .8

C_{12}	C_{13}	C_{20}	C_{21}	C_{22}	C_{23}	C_{31}	C_{40}	p
−.0248000	.0032000	−.1464750	−.1116000	.0036000	.0883500	.0228000	−.0206625	.1
−.0480000	.0064000	−.2400000	−.1920000	.0064000	.1440000	.0384000	−.0336000	.2
−.0696000	.0096000	−.2892750	−.2436000	.0084000	.1725500	.0476000	−.0401625	.3
−.0896000	.0128000	−.3024000	−.2688000	.0096000	.1792000	.0512000	−.0416000	.4
−.1080000	.0160000	−.2868750	−.2700000	.0100000	.1687500	.0500000	−.0390625	.5
−.1248000	.0192000	−.2496000	−.2496000	.0096000	.1456000	.0448000	−.0336000	.6
−.1400000	.0224000	−.1968750	−.2100000	.0084000	.1137500	.0364000	−.0261625	.7
−.1536000	.0256000	−.1344000	−.1536000	.0064000	.0768000	.0256000	−.0176000	.8
−.1656000	.0288000	−.0672750	−.0828000	.0036000	.0379500	.0132000	−.0086625	.9

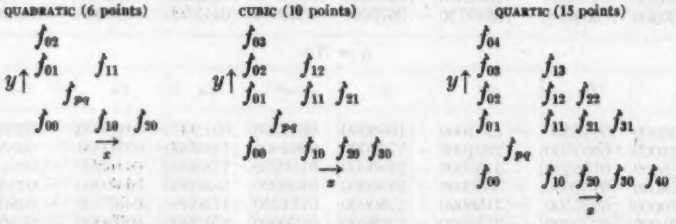
q = .9

C_{12}	C_{13}	C_{20}	C_{21}	C_{22}	C_{23}	C_{31}	C_{40}	p
−.0135000	.0016500	−.1350000	−.1215000	.0020250	.0855000	.0256500	−.0206625	.1
−.0261000	.0033000	−.2204000	−.2088000	.0036000	.1392000	.0432000	−.0336000	.2
−.0378000	.0049500	−.2646000	−.2646000	.0047250	.1666000	.0535500	−.0401625	.3
−.0486000	.0066000	−.2754000	−.2916000	.0054000	.1728000	.0576000	−.0416000	.4
−.0585000	.0082500	−.2600000	−.2925000	.0056250	.1625000	.0562500	−.0390625	.5
−.0675000	.0099000	−.2250000	−.2700000	.0054000	.1400000	.0504000	−.0336000	.6
−.0756000	.0115500	−.1764000	−.2268000	.0047250	.1092000	.0409500	−.0261625	.7
−.0828000	.0132000	−.1196000	−.1656000	.0036000	.0736000	.0288000	−.0176000	.8
−.0891000	.0148500	−.0594000	−.0891000	.0020250	.0363000	.0148500	−.0086625	.9

be wasteful to employ another expression (such as the double-Lagrangian formula).

The present tables will facilitate double interpolation according to (2) by providing the exact numerical values of the coefficients of $f_{s,t} \equiv f(x_s, y_t)$, namely $C_{s,t} \equiv \binom{n-p-q}{n-s-t}\binom{p}{s}\binom{q}{t}$ for all values of s, t arising when $n = 2, 3$, or 4, i.e. in quadratic, cubic, or quartic interpolation. Here n refers to the total degree of the approximating polynomial in p and q. These coefficients $C_{s,t}$ (understood to be functions of n, p, and q) are denoted by A_{st}, B_{st}, and C_{st} in the respective quadratic, cubic, and quartic cases. They are given here, for all values of s and t where $s + t$ ranges from 0 to n, for both p and q each ranging from 0.1 to 0.9 at intervals of 0.1.

The actual points x_s, y_t involved in $f_{s,t} \equiv f(x_s, y_t)$ form a lattice in the pattern of a right-triangle, with the desired point x, y in the corner square. Thus for interpolation of total degree 2, 3, or 4, the values of $f_{s,t}$ correspond to $x_{s,t}$ at the following points: (f_{20} instead of $f_{2,0}$, etc. for convenience)



With the understanding that the coefficients of $f_{s,t}$, i.e. A_{st}, B_{st}, or C_{st}, are functions of p and q, the interpolation formulas assume the following very simple and direct forms:

(4) $f(x + ph_1, y + qh_2) \equiv f_{pq}$

$$= A_{00}f_{00} + A_{01}f_{01} + \cdots + A_{11}f_{11} + A_{20}f_{20} \text{ (Quadratic)},$$

(5) $f(x + ph_1, y + qh_2) \equiv f_{pq} = B_{00}f_{00} + B_{01}f_{01} + \cdots + B_{21}f_{21} + B_{30}f_{30}$ (Cubic),

(6) $f(x + ph_1, y + qh_2) \equiv f_{pq} = C_{00}f_{00} + C_{01}f_{01} + \cdots + C_{31}f_{31} + C_{40}f_{40}$ (Quartic).

It is apparent that these formulas, i.e. (4) – (6), are adapted to rapid and large-scale machine interpolation. The advantage of this method over the use of double differences is apparent from any example. The saving in labor over using double-Lagrangian interpolation, (i.e. first interpolating for f_{iq}, $i = 0, 1, \cdots, n$ and then from the values of f_{iq} to obtain f_{pq}, by using Lagrangian interpolation coefficients for a single variable in each case), is obvious when the amounts of multiplication are compared. Thus the double-Lagrangian method, apart from the separate additions involved, requires 9 points with 12 multiplications for quadratic interpolation, 16 points with 20 multiplications for cubic interpolation, and 25 points with 30 multiplications for quartic interpolation. Of course, in all fairness to the double-Lagrangian method, it must be noted that it holds in the more extended case where $f(x, y)$ behaves as an n-ic in each variable.

The double Gregory-Newton formula is not always the most accurate formula

to use, especially when f_{pq} can be located in a more central position relative to the $f_{s,t}$'s. Yet it was chosen for adaptation according to (2) because it involves the fewest number of points $f_{s,t}$, and even more important, it is the only formula based upon differences that can be used everywhere in the plane, near the beginning or end of the table as a function of p or q, and also near any "corner"; i.e. a combined beginning, or beginning and end, etc. for both p and q. Near a "corner," suitable orientation will always enable the f_{pq} to appear in a left-hand corner square of a right-triangle lattice.

It sometimes happens that either $q = 0$ or $p = 0$, and then the interpolation reduces to that for a single variable in either the x or y direction. For the sake of convenience, the coefficients for $q = 0$ or $p = 0$, although included in published tables of Lagrangian coefficients, are here tabulated separately in Appendix II. The coefficients of the entries f_i where i refers to either the x or y direction, were obtained from the rearrangement of the ordinary Gregory-Newton formula. The coefficients are denoted by A_i, B_i, and C_i for quadratic, cubic, and quartic interpolation respectively. Although their use does not afford accuracy as great as that from coefficients based on a central location for f_{pq}, they have the advantage of being applicable everywhere, including the beginning or end of a table.

APPENDIX I: Alternative Proof that

$$C_{s,t,\cdots,u} = \binom{n-p-q-\cdots-r}{n-s-t-\cdots-u}\binom{p}{s}\binom{q}{t}\cdots\binom{r}{u}.$$

The multiple Gregory-Newton formula of degree n, for a function of ω variables, is

$$(1') \qquad f(x+ph_1, y+qh_2, \cdots, z+rh_\omega) = \sum_{i+j+\cdots+k=0}^{n} \binom{p}{i}\binom{q}{j}\cdots\binom{r}{k} \Delta_x^{i+j+\cdots+k} \Delta_y^{j+\cdots+k} \cdots \Delta_z^{k} f(x, y, \cdots, z).$$

The right member can be expressed as $\sum_{s+t+\cdots+u=0}^{n} C_{s,t,\cdots,u} f_{s,t,\cdots,u}$, where it will be shown that

$$(2') \qquad C_{s,t,\cdots,u} = \binom{n-p-q-\cdots-r}{n-s-t-\cdots-u}\binom{p}{s}\binom{q}{t}\cdots\binom{r}{u}.$$

From (1'), and the fact that

$$\Delta^m f_0 = \sum_{i=0}^{m} (-1)^{i+m}\binom{m}{i} f_i,$$

$$(3') \qquad C_{s,t,\cdots,u} = \sum_{i+j+\cdots+k=s+t+\cdots+u}^{n} \binom{p}{i}\binom{q}{j}\cdots\binom{r}{k}(-1)^{i+s+j+t+\cdots+k+u} \cdot\binom{i}{s}\binom{j}{t}\cdots\binom{k}{u}.$$

Notice that the $i + j + \cdots + k$ cannot be less than $s + t + \cdots + u$ since $i \geqslant s, j \geqslant t, \cdots, k \geqslant u$, which is imposed by the condition that the mixed partial difference of order $i + j + \cdots + k$ involves $f_{s,t,\cdots,u}$.

The proof of (2′) is by induction. For one variable, (3′) becomes

$$C_s = \sum_{i=s}^{n} (-1)^{i+s} \binom{p}{i}\binom{i}{s} = \binom{p}{s} \sum_{i=s}^{n} (-1)^{i-s} \binom{p-s}{i-s} \tag{4′}$$

$$= \binom{p}{s} \sum_{m=0}^{n-s} (-1)^m \binom{p-s}{m}.$$

But since $\Delta_m (-1)^m \binom{a}{m} = (-1)^{m+1} \left\{ \binom{a}{m+1} + \binom{a}{m} \right\} = (-1)^{m+1} \binom{a+1}{m+1}$,

the last sum in (4′) is equal to

$$\left[(-1)^{m-1} \binom{p-s-1}{m-1} \right]_{m=0}^{m=n-s+1},$$

or $(-1)^{n-s} \binom{p-s-1}{n-s} + \binom{p-s-1}{-1}$. But $\binom{p-s-1}{-1} = 0$ if the definition of $\binom{p}{i}$ is extended to negative values of i by means of the gamma function. Since $(-1)^{n-s} \binom{p-s-1}{n-s} = \binom{n-p}{n-s}$, $C_s = \binom{n-p}{n-s}\binom{p}{s}$, which establishes (2′) for one variable.

Now assuming (2′) true for a certain number of variables $x, y, \cdots$, it is proven for one more variable z as follows, (using r, k, and u to denote the tabular fraction, order of difference, and subscript of function, pertaining to the variable z):

From (3′), by rewriting the right member,

$$C_{s,t,\cdots,u} = \sum_{k=u}^{n-s-t-\cdots} (-1)^{k+u} \binom{r}{k}\binom{k}{u} \sum_{i+j+\cdots=n-s+t+\cdots}^{n-k} \tag{5′}$$

$$\cdot (-1)^{i+s+j+t+\cdots} \binom{p}{i}\binom{i}{s}\binom{q}{j}\binom{j}{t} \cdots$$

which, by the hypothesis of the induction, is equal to

$$\binom{p}{s}\binom{q}{t} \cdots \sum_{k=u}^{n-s-t-\cdots} (-1)^{k+u} \binom{r}{k}\binom{k}{u}\binom{n-k-p-q-\cdots}{n-k-s-t-\cdots} \tag{6′}$$

which is

$$\binom{p}{s}\binom{q}{t} \cdots \binom{r}{u} \sum_{k=u}^{n-s-t-\cdots} (-1)^{k-u} \binom{r-u}{k-u}\binom{n-k-p-q-\cdots}{n-k-s-t-\cdots}. \tag{7′}$$

Noting that $(-1)^{k-u} \binom{r-u}{k-u} = \binom{k-r-1}{k-u}$, and setting $k-u=m$,

$$C_{s,t,\cdots,u} = \binom{p}{s}\binom{q}{t} \cdots \binom{r}{u} \sum_{m=0}^{n-s-t-\cdots-u} \tag{8′}$$

$$\cdot \binom{m+u-r-1}{m}\binom{n-p-q-\cdots-u+1-m-1}{n-s-t-\cdots-u-m}.$$

By Newton's backward-difference interpolation formula, when $f(\alpha)$ is a polynomial of the hth degree in α,

$$f(\alpha) = \sum_{m=0}^{h} \binom{m+\alpha}{m} \Delta^m f(-m-1). \tag{9′}$$

Now when

$$f(\alpha) = \binom{\alpha+\beta}{h}, \qquad \Delta f(\alpha) = \binom{\alpha+\beta}{h-1}, \qquad \text{so that } \Delta^m f(\alpha) = \binom{\alpha+\beta}{h-m}.$$

Thus from (9′) it follows that

$$\binom{\alpha+\beta}{h} = \sum_{m=0}^{h} \binom{m+\alpha}{m}\binom{-m-1+\beta}{h-m}. \tag{10′}$$

Identifying h with $n - s - t - \cdots - u$, α with $u - r - 1$, and β with $n - p - q - \cdots - u + 1$, so that $\alpha + \beta = n - p - q - \cdots - r$, the summation in equation (8′) is seen to be precisely $\binom{n-p-q-\cdots-r}{n-s-t-\cdots-u}$. Thus

$$C_{s,t,\cdots,u} = \binom{n-p-q-\cdots-r}{n-s-t-\cdots-u}\binom{p}{s}\binom{q}{t}\cdots\binom{r}{u}$$ for another variable,

which proves it for any number of variables.

APPENDIX II: TABLES OF COEFFICIENTS FOR $q = 0$ or $p = 0$.

Quadratic

p (or q)	A_0	A_1	A_2
.1	.855	.190	−.045
.2	.720	.360	−.080
.3	.595	.510	−.105
.4	.480	.640	−.120
.5	.375	.750	−.125
.6	.280	.840	−.120
.7	.195	.910	−.105
.8	.120	.960	−.080
.9	.055	.990	−.045

Cubic

p (or q)	B_0	B_1	B_2	B_3
.1	.8265	.2755	−.1305	.0285
.2	.6720	.5040	−.2240	.0480
.3	.5355	.6885	−.2835	.0595
.4	.4160	.8320	−.3120	.0640
.5	.3125	.9375	−.3125	.0625
.6	.2240	1.0080	−.2880	.0560
.7	.1495	1.0465	−.2415	.0455
.8	.0880	1.0560	−.1760	.0320
.9	.0385	1.0395	−.0945	.0165

Quartic

p (or q)	C_0	C_1	C_2	C_3	C_4
.1	.8058375	.3581500	−.2544750	.1111500	−.0206625
.2	.6384000	.6384000	−.4256000	.1824000	−.0336000
.3	.4953375	.8491500	−.5244750	.2201500	−.0401625
.4	.3744000	.9984000	−.5616000	.2304000	−.0416000
.5	.2734375	1.0937500	−.5468750	.2187500	−.0390625
.6	.1904000	1.1424000	−.4896000	.1904000	−.0336000
.7	.1233375	1.1511500	−.3984750	.1501500	−.0261625
.8	.0704000	1.1264000	−.2816000	.1024000	−.0176000
.9	.0298375	1.0741500	−.1464750	.0511500	−.0086625

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